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Blossoming beyond Extended Chebyshev Spaces

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In a previous series of papers a theory of blossoming was developed for spaces of functions on an interval I spanned by the constant functions and functions $\Phi_1, ..., \Phi_n$, where $\Phi'_1, ..., \Phi'_n$ span an extended Chebyshev space. This theory was then used to construct a generalisation of the Bernstein basis and the de Casteljau algorithm. Also considered were functions defined to be piecewise in such spaces, leading to generalisations of B-splines and the de Boor algorithm. Here we relax the condition that $\Phi'_1, ..., \Phi'_n$ span an extended Chebyshev space, while retaining all the nice properties of the earlier theory. This allows us to include a large variety of new spaces, including spaces of polynomials which have been found to be successful for tension methods for shape-preserving interpolation. © 2001 Academic Press

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1. INTRODUCTION

In computer-aided design it is extremely useful to design curves using the space $\mathscr{P}_n$ of polynomials of degree n, employing the Bernstein basis, Bézier points, and the de Casteljau algorithm. More generally we can use spline functions, employing the B-spline basis and the de Boor algorithm. An elegant tool for unifying and clarifying these concepts is the notion of blossoming [20]. In a series of papers [13, 15, 18, 19], these ideas have been extended from $\mathscr{P}_n$ to a much larger class of spaces. Any such space is spanned by the constant functions and functions $\Phi_1, ..., \Phi_n$ on an interval I, where $\Phi'_1, ..., \Phi'_n$ span an extended Chebyshev space. A generalisation of

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the notion of blossoming is then used to derive generalisations of the Berntein and B-spline bases with de Casteljau and de Boor algorithms and optimal shape preserving properties.

A different type of generalisation of $\mathscr{P}_n$ was considered first by Costantini in [3], then by Kaklis and Pandelis in [10]. Motivated by tension methods for shape-preserving interpolation, they considered the space spanned by the functions $\{1, x, x^{2+m}, (1-x)^{2+m}\}$ on $[0, 1]$, where m is any positive integer. This space has proved fruitful for shape-preserving interpolation methods (see [5], and the survey [4]). However it does not fall under the theory discussed above because the functions $\{1, x^{1+m}, (1-x)^{1+m}\}$ do not span an extended Chebyshev space on $[0, 1]$ (unless $m=1$), since functions in the space may have too many zeros at 0 or 1.

The purpose of this paper is to extend the theory discussed in the first paragraph to include the space considered by Costantini and by Kaklis and Pandelis. More generally it includes the space spanned by the functions $\{1, x, ..., x^{n-2}, x^{n-1+m_1}, (1-x)^{n-1+m_2}\}$ on $[0,1]$, where n, m_1, m_2 are positive integers, which reduces to $\mathscr{P}_n$ when $m_1=m_2=1$. This case is discussed in detail in Section 4, where examples are given to show how m_1 and m_2 act as tension parameters.

The general theory is developed in Section 2. In Section 2.1 we define the notion of blossoming without the previously required condition that the vectors $\Phi^{(j)}(x)=(\Phi_1^{(j)}(x), ..., \Phi_n^{(j)}(x))^T$, $j=1, ..., n$, are linearly independent for all x in I. Under the much weaker condition that $\Phi'(x), ..., \Phi^{(n-1)}(x)$, $\Phi^{(s)}(x)$ are linearly independent for some $s \geqslant n$, it is shown in the rest of Section 2 that Bernstein and B-spline bases can still be defined with the usual properties including de Casteljau and de Boor algorithms. Then in Section 3 we construct a very large class of spaces satisfying the theory of Section 2 by considering kernels of differential operators associated with weight functions, generalising a classical construction of extended Chebyshev spaces. The spaces discussed in Section 4 are special cases of this construction.

2. QUASI-CHEBYSHEV FUNCTIONS AND BLOSSOMING

Throughout the paper, given a function Φ defined on a real interval I, with values in an affine space, the osculating flat of order ℓ of Φ at a point x (at which Φ is ℓ times differentiable) denotes the affine flat passing through $\Phi(x)$ and the direction of which is the linear space spanned by its first ℓ derivatives at x. We denote it by $\operatorname{Osc}_\ell \Phi(x)$, so that:

$$\operatorname{Osc}_\ell \Phi(x) := \{\Phi(x)+\lambda_1 \Phi'(x)+ \cdots +\lambda_\ell \Phi^{(\ell)}(x) \mid \lambda_1, ..., \lambda_\ell \in \mathbb{R}\}.$$

For simplicity, assume now that $\Phi = (\Phi_1, ..., \Phi_n)^T \colon I \to \mathbb{R}^n$ is infinitely many times differentiable. In [13], such a function is said to be a *Chebyshev function of order n on I* when the space spanned by the n functions $\Phi'_1, ..., \Phi'_n$ is an n-dimensional *extended Chebyshev space on I*, i.e., when any nonzero element of this space has at most $(n-1)$ zeros (counted with multiplicities) in I. This property was proved to be equivalent to the following two simultaneous ones ([18]):

(1) for all $x \in I$, the n derivative vectors $\Phi'(x), ..., \Phi^{(n)}(x)$ are linearly independent,

(2) for all distinct $\tau_1, ..., \tau_r \in I$ and for all positive integers $\mu_1, ..., \mu_r$ such that $\sum_{i=1}^r \mu_i = n$, the r osculating flats $\mathrm{Osc}_{n-\mu_i} \Phi(\tau_i)$ intersect at a single point.

Whenever the two n-tuples $(x_1, ..., x_n)$ and $(\underbrace{\tau_1, ..., \tau_1}_{\mu_1 \text{ times}}, ..., \underbrace{\tau_r, ..., \tau_r}_{\mu_r \text{ times}})$ are equal up to a permutation, the latter single point is labelled as $\varphi(x_1, ..., x_n)$. This provides a function $\varphi \colon I^n \to \mathbb{R}^n$ called the *blossom* of Φ. The blossom satisfies the following three properties:

(i) φ is symmetric on I^n,

(ii) $\varphi(x, ..., x) = \Phi(x)$ for all $x \in I$,

(iii) φ is *pseudo-affine* with respect to each variable, in the sense that, for any $x_1, ..., x_{n-1} \in I$, the point $\varphi(x_1, ..., x_{n-1}, x)$, $x \in I$, varies in a strictly monotonic way along an affine line.

These three properties are essential for developing the basic algorithms of geometric design. The first two ones are obvious to obtain from the definition, the third one was proved in [18].

Let us now focus on the elementary case $n = 2$. Then, $\Phi = (\Phi_1, \Phi_2)^T \colon I \to \mathbb{R}^2$ is a Chebyshev function of order 2 on I iff:

$$\text{for all } x \in I,\ \Phi'(x), \Phi''(x) \text{ are linearly independent,} \tag{2.1}$$

$$\text{for all } x, y \in I,\ x \neq y,\ \Phi'(x), \Phi'(y) \text{ are linearly independent.} \tag{2.2}$$

If (2.1) and (2.2) are satisfied, then the blossom $\varphi = (\varphi_1, \varphi_2)$ of Φ is defined by:

$$\{\varphi(x, y)\} := \mathrm{Osc}_1\, \Phi(x) \cap \mathrm{Osc}_1\, \Phi(y), \qquad \varphi(x, x) := \Phi(x), \tag{2.3}$$

for all distinct $x, y \in I$. Consider the function $\Phi(x) := (x, x^4)^T$, $x \in \mathbb{R}$. Then, for all $x, y \in \mathbb{R}$,

$$\det(\Phi'(x), \Phi''(x)) = 12x^2, \qquad \det(\Phi'(x), \Phi'(y)) = 4(y-x)(x^2 + xy + y^2). \tag{2.4}$$

Equalities (2.4) show that Φ is a Chebyshev function of order 2 on $]0, +\infty[$. According to (2.3), one can easily prove that its blossom on $]0, +\infty[\times]0, +\infty[$ is given by:

$$\varphi_1(x, y) = \frac{3}{4}\,\frac{x^3 + x^2 y + x y^2 + y^3}{x^2 + xy + y^2}, \qquad \varphi_2(x, y) = 3\,\frac{x^3 y^3}{x^2 + xy + y^2}. \tag{2.5}$$

On the other hand, due to the first part of (2.4), Φ cannot be a Chebyshev function on any interval I containing 0. Nevertheless, due to the second part of (2.4), property (2.2) is satisfied on the whole real line. Therefore the two tangent lines $\mathrm{Osc}_1\,\Phi(x)$ and $\mathrm{Osc}_1\,\Phi(y)$ do intersect at a single point whatever the points $x, y \in \mathbb{R}$, $x \neq y$. This observation makes it natural to define the blossom φ by formulae (2.3) not only on $]0, +\infty[^2$, but even on $\mathbb{R}^2$. Explicitly, $\varphi(x, y)$ is then given by (2.5) for all $(x, y) \in \mathbb{R}^2 \backslash \{(0, 0)\}$ and by $\varphi(0, 0) := (0, 0)$. The strict convexity of the parametric curve defined by Φ guarantees that, for any fixed $a \in \mathbb{R}$, the point $\varphi(a, x)$, $x \in \mathbb{R}$, varies in a strictly monotonic way along the tangent line $\mathrm{Osc}_1\,\Phi(a)$. Since properties (i) and (ii) mentioned above are obviously valid on $\mathbb{R}$, the blossom φ satisfies all expected properties on $\mathbb{R}^2$.

This very simple example suggests that, more generally, it might be possible to develop the blossoming principle without assuming the linear independence of the first n derivatives, that is beyond the strict framework of extended Chebyshev spaces. This is the problem we will address in this section.

2.1. *Quasi-Chebyshev Functions*

Throughout the section, $\mathscr{A}$ denotes a given n-dimensional real affine space and I a real interval. From now on, inside a tuple, the notation x^μ will mean $\underbrace{x, ..., x}_{\mu \text{ times}}$.

Definition 2.1. Consider a function $\Phi: I \to \mathscr{A}$. If $n \geqslant 2$, Φ will be said to be a *quasi-Chebyshev function of order n on I* if it is C^{n-1} on I and, if, for all distinct $\tau_1, ..., \tau_r \in I$, and all positive integers $\mu_1, ..., \mu_r$ such that $\sum_{i=1}^r \mu_i = n$, the intersection $\bigcap_{i=1}^r \mathrm{Osc}_{n-\mu_i}\,\Phi(\tau_i)$ consists of a single point. If $n = 1$, Φ will be said to be a quasi-Chebyshev function of order 1 on I if it is C^0 on I and if the point $\Phi(x)$, $x \in I$, varies strictly monotonically along the affine line $\mathscr{A}$.

Then, the function $\varphi: I^n \to \mathscr{A}$ defined by:

$$\{\varphi(x_1, ..., x_n)\} := \bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i}\,\Phi(\tau_i), \tag{2.6}$$

for all n-tuples $(x_1, ..., x_n) \in I^n$ which are equal to $(\tau_1^{\mu_1}, ..., \tau_r^{\mu_r})$ up to a permutation, will be called *the blossom* of Φ.

A few comments on these definitions. Assuming that $n \geqslant 2$, differentiability of order $n-1$ is the minimum assumption to require in order to give sense to all the osculating flats possibly involved in (2.6). Of course, when $r=1$, then $\mu_1 = n$ and the corresponding intersection reduces to $\mathrm{Osc}_0\, \Phi(\tau_1)$, which always consists of the single point $\Phi(\tau_1)$. We could therefore have required r to be greater than or equal to 2 in Definition 2.1. It also results from the latter observation that, when restricted to the diagonal of I^n, the blossom φ of a quasi-Chebyshev function Φ of order n gives the function Φ itself. Moreover, φ is by nature a symmetric function on I^n.

PROPOSITION 2.2. *Assume that $\Phi\colon I \to \mathscr{A}$ is a quasi-Chebyshev function of order n on I $(n \geqslant 2)$. Then, given any two distinct points $a, b \in I$, and any integer i, $0 < i < n$, the n derivative vectors $\Phi'(a), ..., \Phi^{(i)}(a), \Phi'(b), ..., \Phi^{(n-i)}(b)$ are linearly independent.*

Proof. From Definition 2.1, we know the existence of the point

$$\{\varphi(a^{n-i}, b^i)\} := \mathrm{Osc}_i\, \Phi(a) \cap \mathrm{Osc}_{n-i}\, \Phi(b).$$

This implies the existence of unique real numbers $\lambda_1, ..., \lambda_i, \mu_1, ..., \mu_{n-i}$ such that

$$\Phi(b) - \Phi(a) = \lambda_1\, \Phi'(a) + \cdots + \lambda_i\, \Phi^{(i)}(a) + \mu_1\, \Phi'(b) + \cdots + \mu_{n-i}\, \Phi^{(n-i)}(b),$$

which proves that the n vectors $\Phi'(a), ..., \Phi^{(i)}(a), \Phi'(b), ..., \Phi^{(n-i)}(b)$ form a basis of the direction of the affine space $\mathscr{A}$. ∎

COROLLARY 2.3. *Suppose that $\Phi\colon I \to \mathscr{A}$ is a quasi-Chebyshev function of order n on I. Then, given any point $a \in I$, and any integer $\ell \leqslant n-1$, the osculating flat $\mathrm{Osc}_\ell\, \Phi(a)$ is ℓ-dimensional. Moreover,*

$$\Phi(x) \notin \mathrm{Osc}_{n-1}\, \Phi(a) \qquad \textit{for} \quad x \in I \backslash \{a\}. \tag{2.7}$$

In particular, the affine space $\mathrm{aff}(\mathrm{Im}\, \Phi)$ *spanned by the image of Φ is equal to $\mathscr{A}$.*

Proof. For $n=1$, (2.7) results from the strict monotonicity of Φ. Suppose $n \geqslant 2$. It follows from Proposition 2.2 that the first $n-1$ derivatives $\Phi'(a), ..., \Phi^{(n-1)}(a)$ are linearly independent at any point $a \in I$, whence the result on the dimension of all osculating flats of order at most $n-1$. Fixing the point $a \in I$, the function N defined by

$$N(x) := \det[\Phi'(a), ..., \Phi^{(n-1)}(a), \Phi(x) - \Phi(a)], \qquad x \in I, \tag{2.8}$$

is C^{n-1} on I like Φ. According to Proposition 2.2, its first derivative

$$N'(x) = \det[\Phi'(a), ..., \Phi^{(n-1)}(a), \Phi'(x)],$$

vanishes only for $x = a$. Since $N(a) = 0$, Rolle's theorem allows us to conclude that $N(x) \neq 0$ for $x \neq a$, which proves (2.7). Since $\mathrm{Osc}_{n-1}\, \Phi(a)$ is an $(n-1)$-dimensional affine subspace of $\mathscr{A}$, the equality $\mathrm{aff}(\mathrm{Im}\ \Phi) = \mathscr{A}$ follows. ▮

Picking an affine frame $A_0, ..., A_n$ in $\mathscr{A}$, any function $\Phi: I \to \mathscr{A}$ can be written as:

$$\Phi(x) = \sum_{i=0}^{n} \Phi_i(x)\ A_i, \qquad \sum_{i=0}^{n} \Phi_i(x) = 1, \qquad x \in I. \tag{2.9}$$

In the following, we shall denote by $\mathscr{E}(\Phi)$ the associated space $\mathscr{E}(\Phi) := \mathrm{span}(\Phi_0, \Phi_1, ..., \Phi_n) = \mathrm{span}(\mathbf{1}, \Phi_1, ..., \Phi_n)$, which is independent of the chosen frame. Moreover, we call *$\mathscr{E}(\Phi)$-functions* all functions with values in a finite dimensional affine space, all the coordinates of which (w.r.t. to any frame) belong to the space $\mathscr{E}(\Phi)$.

Suppose that Φ is a quasi-Chebyshev function of order n on I. Then, $\mathrm{aff}(\mathrm{Im}\ \Phi)$ being n-dimensional (Corollary 2.3), the associated space $\mathscr{E}(\Phi)$ is $(n+1)$-dimensional. Any $\mathscr{E}(\Phi)$-function F being the image $F = h \circ \Phi$ of Φ under a unique affine map h, its blossom f will naturally be defined by $f := h \circ \varphi$.

Examples 2.4. (i) Let us examine the particular case when $n = 2$. Then a necessary and sufficient condition for a C^1 function $\Phi: I \to \mathscr{A}$ to be a quasi-Chebyshev function is that the determinant $\det[\Phi'(x), \Phi'(y)]$ never vanishes on $I^2 \backslash \{(x, x) \mid x \in I\}$. When the latter property holds, an argument of continuity proves $\det[\Phi'(x), \Phi'(y)]$ to keep a constant strict sign for $x, y \in I$, $x < y$. Consequently, the planar parametric curve defined by Φ is a strictly convex one. It easily follows that, for a fixed $a \in I$, the point $\varphi(a, x)$ varies in a strictly monotonic way along the tangent line $\mathrm{Osc}_1\, \Phi(a)$. According to Definition 2.1, $\varphi(a, .)$ is thus a quasi-Chebyshev function of order 1 on I.

(ii) Given three positive integers m, p, q, consider the function $\Phi: \mathbb{R} \to \mathbb{R}^3$ defined by $\Phi(x) := (x^m, x^{m+p}, x^{m+p+q})^T$. We know that Φ is a

Chebyshev function of order 3 on $]0,+\infty[$ (see [14]). Now, none of the vectors $\Phi'(0)$, $\Phi''(0)$, $\Phi'''(0)$ is equal to zero iff $m+p+q=3$. Consequently, as long as $m+p+q>3$, Φ cannot be a Chebyshev function on any interval containing 0. The condition $m+p+q=3$ means that $\Phi(x)=(x, x^2, x^3)^T$, in which case Φ is a Chebyshev function on the whole real line and we are just working with the ordinary blossoms in the space $\mathscr{E}(\Phi)=\mathscr{P}_3$ of polynomials of degree at most 3.

On the other hand, the two vectors $\Phi'(0)$, $\Phi''(0)$ are linearly independent iff $m=p=1$. Suppose now $\Phi(x)=(x, x^2, x^{q+2})^T$, with $q>1$. Then, one can check that, for any distinct positive x, y, $\mathrm{Osc}_2\Phi(0)\cap\mathrm{Osc}_2\Phi(x)\cap\mathrm{Osc}_2\Phi(y)$, $\mathrm{Osc}_2\Phi(0)\cap\mathrm{Osc}_1\Phi(x)$, and $\mathrm{Osc}_1\Phi(0)\cap\mathrm{Osc}_2\Phi(x)$ all consist of a single point. Therefore, Φ is a quasi-Chebyshev function on $[0,+\infty[$. We will see in the next section that, when $q=2k+1$, Φ is in fact a quasi-Chebyshev function on the whole real line. This is no longer true when $q=2k$. For instance, if $\Phi(x)=(x, x^2, x^4)^T$, it is easy to check that, for all $x\neq 0$, $\Phi(-3x)\in\mathrm{Osc}_2\,\Phi(x)$, which contradicts property (2.7) on $\mathbb{R}$.

2.2. *Subblossoms*

In this subsection we consider a given quasi-Chebyshev function Φ of order $n\geqslant 2$ on I. Then, fixing a point a in I, we define

$$\tilde{\Phi}(x):=\varphi(a, x^{n-1}), \qquad x\in I. \tag{2.10}$$

THEOREM 2.5. *Assume Φ to be C^∞ on I. Let us suppose that there exists an integer $s\geqslant n$ such that the n derivative vectors $\Phi'(a), \ldots, \Phi^{(n-1)}(a)$, $\Phi^{(s)}(a)$ are linearly independent. Then $\tilde{\Phi}$ is a C^∞ quasi-Chebyshev function of order $n-1$ on I with values in $\mathrm{Osc}_{n-1}\,\Phi(a)$. Its blossom $\tilde{\varphi}$ is defined by:*

$$\tilde{\varphi}(x_1, \ldots, x_{n-1})=\varphi(x_1, \ldots, x_{n-1}, a), \qquad x_1, \ldots, x_{n-1}\in I. \tag{2.11}$$

Proof. We can suppose that $n\geqslant 3$ since for $n=2$ the result was proved in Example 2.4, (i). The proof is built according to the same idea as in the case of Chebyshev functions (see [15,18]). However the slight differences due to the missing linear independence of the first n derivatives at each point of I requires us to give it again. It includes several steps.

(1) *Let us first prove that $\tilde{\Phi}$ is C^∞ on I.* Due to the definition of $\tilde{\Phi}$, we have:

$$\tilde{\Phi}(a)=\Phi(a), \qquad \{\tilde{\Phi}(x)\}=\mathrm{Osc}_{n-1}\,\Phi(a)\cap\mathrm{Osc}_1\,\Phi(x) \text{ if } x\in I\backslash\{a\}. \tag{2.12}$$

Consequently, there exists a function $\lambda\colon I\to\mathbb{R}$ such that:

$$\tilde{\Phi}(x)=\Phi(x)+\lambda(x)\,\Phi'(x), \qquad x\in I. \tag{2.13}$$

In order to prove that $\tilde{\Phi}$ is C^∞ on I, it is therefore sufficient (and actually it is also necessary) to prove that function λ is C^∞ on I. Now, from (2.12) we can derive that:

$$\lambda(x) = \begin{cases} 0 & \text{if} \quad x = a, \\ -\dfrac{N(x)}{N'(x)} & \text{if} \quad x \neq a, \end{cases} \tag{2.14}$$

where N is defined as in (2.8), i.e., $N(x) := \det[\Phi'(a), ..., \Phi^{(n-1)}(a), \Phi(x) - \Phi(a)]$. This function N is C^∞ on I and for all $i \geqslant 1$, $N^{(i)}(x) = \det[\Phi'(a), ..., \Phi^{(n-1)}(a), \Phi^{(i)}(x)]$. Therefore, according to Proposition 2.2, the first derivative of N never vanishes on $I \backslash \{a\}$. Hence, λ is clearly C^∞ on $I \backslash \{a\}$. On the other hand, we have:

$$N(a) = N'(a) = \cdots = N^{(n-1)}(a) = 0, \; N^{(s)}(a) \neq 0. \tag{2.15}$$

From (2.15) one can prove that λ is in fact C^∞ on the whole interval I. This follows from the following lemma we state here without proof (see [15]).

LEMMA 2.6. *Let J be a real interval containing a. Suppose that f: $J \to \mathbb{R}$ is C^∞ on J and satisfies $f(a) = f'(a) = \cdots = f^{(k-1)}(a) = 0$, $f^{(k)}(a) \neq 0$, and $f'(t) \neq 0$ for all $t \in J \backslash \{a\}$. Then the function g defined on J by*

$$g(t) = \frac{f(t)}{f'(t)} \qquad \textit{if} \quad t \neq 0, \qquad g(a) = 0,$$

is C^∞ on J and $g'(a) = 1/k$.

(2) *Let us determine the osculating flats of $\tilde{\Phi}$.* By differentiation of relation (2.13) up to order $i \geqslant 0$, we obtain:

$$\begin{aligned} \tilde{\Phi}^{(i)}(x) &= \lambda(x)\, \Phi^{(i+1)}(x) + (1 + i\lambda'(x))\, \Phi^{(i)}(x) \\ &\quad + \sum_{\ell=0}^{i-2} \binom{i}{\ell} \lambda^{(i-\ell)}(x)\, \Phi^{(\ell+1)}(x), \qquad x \in I. \end{aligned} \tag{2.16}$$

This proves that, for all $i \geqslant 0$ and all $x \in I$, $\mathrm{Osc}_i\, \tilde{\Phi}(x) \subset \mathrm{Osc}_{i+1} \Phi(x)$. Since function $\tilde{\Phi}$ takes its values in $\mathrm{Osc}_{n-1} \Phi(a)$, we thus have, for all $x \in I$:

$$\mathrm{Osc}_i\, \tilde{\Phi}(x) \subset \mathrm{Osc}_{i+1}\, \Phi(x) \cap \mathrm{Osc}_{n-1}\, \Phi(a), \qquad i \geqslant 0. \tag{2.17}$$

Suppose first that $x \neq a$. For $i \leqslant n-2$, we know from Corollary 2.3 that $\mathrm{Osc}_{i+1}\, \Phi(x)$ is $(i+1)$-dimensional. Furthermore, by (2.7), $\Phi(x) \notin \mathrm{Osc}_{n-1}\, \Phi(a)$, hence $\mathrm{Osc}_{i+1}\, \Phi(x) \not\subset \mathrm{Osc}_{n-1}\, \Phi(a)$. Accordingly, the right hand side of (2.17) is of dimension at most i. On the other hand, since $N(x) \neq 0$, formula (2.14) proves that $\lambda(x) \neq 0$. We can thus derive from (2.16) that the osculating space $\mathrm{Osc}_i\, \tilde{\Phi}(x)$ is i-dimensional. Due to (2.17), the latter observations prove that:

$$\mathrm{Osc}_i\, \tilde{\Phi}(x) = \mathrm{Osc}_{i+1}\, \Phi(x) \cap \mathrm{Osc}_{n-1}\, \Phi(a), \qquad x \neq a,\ i = 0, ..., n-2. \tag{2.18}$$

Suppose now that $x = a$. Since $\lambda(a) = 0$, equality (2.16) then reduces to:

$$\tilde{\Phi}^{(i)}(a) = (1 + i\lambda'(a))\, \Phi^{(i)}(a) + \sum_{\ell=0}^{i-2} \binom{i}{\ell} \lambda^{(i-\ell)}(a)\, \Phi^{(\ell+1)}(a), \qquad i \geqslant 0. \tag{2.19}$$

According to Lemma 2.3, $\lambda'(a) = -1/k$, where $k \geqslant n$ is the smallest integer such that $N^{(k)}(a) \neq 0$, i.e., such that $\Phi'(a), ..., \Phi^{(n-1)}(a), \Phi^{(k)}(a)$ are linearly independent. Consequently, $(1 + i\lambda'(a)) \neq 0$ for all $i \leqslant n-1$. It clearly follows that

$$\mathrm{Osc}_i\, \tilde{\Phi}(a) = \mathrm{Osc}_i\, \Phi(a), \qquad i = 0, ..., n-1. \tag{2.20}$$

(3) *Let us prove that* $\tilde{\Phi}$ *is a quasi-Chebyshev function of order* $n-1$ *on* I. Choose any distinct $\tau_1, ..., \tau_r \in I$ and any positive integers $\mu_1, ..., \mu_r$ such that $\sum_{i=1}^{r} \mu_i = n-1$. Then, using equalities (2.18) and (2.20), one can check that:

$$\bigcap_{i=1}^{r} \mathrm{Osc}_{n-1-\mu_i}\, \tilde{\Phi}(\tau_i) = \{\varphi(\tau_1{}^{\mu_1}, ..., \tau_r{}^{\mu_r}, a)\}. \tag{2.21}$$

Equality (2.21) proves both that $\tilde{\Phi}$ is a quasi-Chebyshev function of order $n-1$ on I and that its blossom $\tilde{\varphi}$ satisfies (2.11). For more details, we refer to [15]. ∎

Remark 2.7. The tricky part of the proof is in fact the differentiability of function $\tilde{\Phi}$, or, equivalently, that of the function λ defined in (2.14). Requiring the n derivative vectors $\Phi'(a), ..., \Phi^{(n-1)}(a), \Phi^{(s)}(a)$ to be linearly independent for some $s \geqslant n$ (which is automatically satisfied, with $s = n$, when Φ is a Chebyshev function on I) is a reasonable assumption to guarantee this differentiability. We note that the assumption holds if the components of Φ are analytic at a. It is intended to prevent the point a to be a zero of infinite multiplicity of the function N defined in (2.8), which might leave the possibility of the ratio N/N' not being continuous at 0. Whether or not it is possible to find a quasi-Chebyshev function Φ for which the latter situation occurs is still an open problem.

However, the result stated in Theorem 2.5 may be valid even though the vectors $\Phi'(a), ..., \Phi^{(n-1)}(a), \Phi^{(s)}(a)$ are linearly dependent for all $s \geqslant n$, as pointed out in the following example. Consider the function Φ defined by:

$$\Phi(x) := (x, x^2, e^{-1/x})^T \qquad \text{if} \quad x \neq 0, \Phi(0) = (0, 0, 0)^T.$$

Since $(e^{-1x})''$ is strictly monotonic on $I := [0, (3-\sqrt{3})/6]$, Theorem 3.1 of the next section will enable us to prove that Φ is a quasi-Chebyshev function of order 3 on I. Choosing $a := 0$, for all integers $s \geqslant 3$, we have $\Phi^{(s)}(a) = 0$. On the other hand, $N(x) := \det(\Phi'(0), \Phi''(0), \Phi(x) - \Phi(0)) = 2\, e^{-1/x}$ for $x \neq 0$. It follows that $\lambda(x) = -x^2$ for all $x \in I$. Therefore function λ is C^∞ on I.

2.3. *Chebyshev-Bernstein Basis*

In this subsection, we shall assume that Φ is a C^∞ quasi-Chebyshev function of order n on I and that it satisfies the following assumption:

> (H_n) for any point $x \in I$, there exists an integer $s \geqslant n$ such that the n derivative vectors $\Phi'(x), ..., \Phi^{(n-1)}(x), \Phi^{(s)}(x)$ are linearly independent.

In order to make it possible to iterate the subblossoming principle, we first need to establish the following elementary result.

LEMMA 2.8. *The quasi-Chebyshev function* $\tilde{\Phi}$ *of order* $n-1$ *defined in* (2.10) *satisfies* (H_{n-1}).

Proof. Consider first a point $x \neq a$ and suppose that $\Phi'(x), ..., \Phi^{(n-1)}(x), \Phi^{(s)}(x)$ are linearly independent. Then, since $\lambda(x) \neq 0$, it easily follows from (2.16) that $\tilde{\Phi}'(x), ..., \tilde{\Phi}^{(n-2)}(x), \tilde{\Phi}^{(s-1)}(x)$ are linearly independent in turn. On the other hand, (2.20) proves the linear independence of the $n-1$ vectors $\tilde{\Phi}'(a), ..., \tilde{\Phi}^{(n-2)}(a), \tilde{\Phi}^{(n-1)}(a)$. ∎

We are now in a position to iterate the subblossoming principle, which will later allow us to develop a de Casteljau type algorithm.

THEOREM 2.9. *For all* $x_1, ..., x_{n-1}, a, b \in I$, $a \neq b$, *there exists a* C^∞ *strictly monotone function* β (*depending on* $x_1, ..., x_{n-1}, a, b$) *such that*:

$$\varphi(x_1, ..., x_{n-1}, x) = [1-\beta(x)]\, \varphi(x_1, ..., x_{n-1}, a) + \beta(x)\, \varphi(x_1, ..., x_{n-1}, b), \qquad x \in I. \tag{2.22}$$

Proof. Lemma 2.8 allows us to iterate the subblossoming principle. It follows that the function $\varphi(x_1, ..., x_{n-1}, .)$ is a C^∞ quasi-Chebyshev function of order 1 on I, whence the result. ∎

From now on, we will suppose that the two points a, b are fixed, and we will apply (2.22) with $(x_1, ..., x_{n-1}) := (a^{n-k-i}, b^i, x^{k-1})$, where x is any given point in I, and k, i are any two integers satisfying $1 \leqslant k \leqslant n$, $0 \leqslant i \leqslant n-k$. The corresponding function β (which depends on x, a, b, k, i) satisfies $\beta(a)=0$, $\beta(b)=1$, and $\beta(y)>0$ when y is located strictly between a and b. Setting $\beta_{i,k}(x) := \beta(x)$, we thus obtain:

$$\varphi(a^{n-k-i}, b^i, x^k) = [1-\beta_{i,k}(x)]\, \varphi(a^{n-k-i+1}, b^i, x^{k-1}) + \beta_{i,k}(x)\, \varphi(a^{n-k-i}, b^{i+1}, x^{k-1}), \qquad x \in I. \tag{2.23}$$

Moreover the function $\beta_{i,k}$ defined in such a way satisfies

$$\beta_{i,k}(a)=0, \quad \beta_{i,k}(b)=1, \quad \beta_{i,k}(x)>0 \qquad \text{for } x \text{ strictly between } a \text{ and } b. \tag{2.24}$$

For any $x \in I$, equalities (2.23) allow the computation in n steps of $\Phi(x) = \varphi(x^n)$ as an affine combination of the $n+1$ starting points

$$\Pi_i := \varphi(a^{n-i}, b^i) \in \mathscr{A}, \qquad i=0, ..., n, \tag{2.25}$$

called the *Chebyshev-Bézier* points of Φ with respect to (a, b). This can be written as follows:

$$\Phi(x) = \sum_{i=0}^{n} \mathscr{B}_i(x)\, \Pi_i, \qquad \sum_{i=0}^{n} \mathscr{B}_i(x) = 1, \qquad x \in I. \tag{2.26}$$

Furthermore, due to (2.24), when x is located strictly between a and b, the latter affine combination is actually a strictly convex one, which means that

$$\mathscr{B}_i(x) > 0 \qquad \text{for } x \text{ strictly between } a \text{ and } b,\ i=0, ..., n. \tag{2.27}$$

The proposition below states additional properties of the points Π_i and the functions $\mathscr{B}_i$. They are slighty different from those obtained in [16].

THEOREM 2.10. *The Chebyshev-Bézier points $\Pi_0, ..., \Pi_n$ of Φ are affinely independent, and satisfy*

$$\begin{aligned} \operatorname{Osc}_i \Phi(a) &= \operatorname{aff}(\Pi_0, ..., \Pi_i), \\ \operatorname{Osc}_i \Phi(b) &= \operatorname{aff}(\Pi_{n-i}, ..., \Pi_n), \qquad i=0, ..., n-1. \end{aligned} \tag{2.28}$$

The functions $\mathscr{B}_i$, $i=0, ..., n$, form a basis of the space $\mathscr{E}(\Phi)$ (called its Chebyshev-Bernstein basis w.r.t. to (a, b)). For $i=1, ..., n-1$, $\mathscr{B}_i$ vanishes exactly i times at a and exactly $n-i$ times at b, while $\mathscr{B}_0$ vanishes at least n times at b and $\mathscr{B}_n$ at least n times at a.

Proof. According to Corollary 2.3, the affine space spanned by the image of Φ is n-dimensional. The affine independence of the Chebyshev-Bézier points $\Pi_0, ..., \Pi_n$ therefore results from (2.26). The fact that $(\mathscr{B}_0, ..., \mathscr{B}_n)$ form a basis of the $(n+1)$-dimensional space $\mathscr{E}(\Phi)$ results by taking the images of the first equality in (2.26) under all real valued affine maps. Since $\Phi(a)=\Pi_0$, due to the affine independence of the Chebyshev-Bézier points, equality (2.27) proves that

$$\mathscr{B}_0(a)=1, \quad \mathscr{B}_i(a)=0 \qquad \text{for} \quad i=1, ..., n. \tag{2.29}$$

On the other hand, differentiation of the two equalities in (2.26) leads to:

$$(\Phi'(a), ..., \Phi^{(n-1)}(a))^T = \mathscr{M}.\,(\Pi_1-\Pi_0, ..., \Pi_n-\Pi_0)^T, \tag{2.30}$$

where $\mathscr{M}$ is the $(n-1)\times n$ matrix defined by $\mathscr{M}_{i,j}=\mathscr{B}_j^{(i)}(a)$, $1\leqslant i\leqslant n-1$, $1\leqslant j\leqslant n$. Now, due to definition (2.25), for $0\leqslant \ell\leqslant n-1$, the $(\ell+1)$ Chebyshev-Bézier points $\Pi_0, ..., \Pi_\ell$ belong to $\mathrm{Osc}_\ell\,\Phi(a)$. Therefore, $(\Pi_1-\Pi_0, ..., \Pi_\ell-\Pi_0)$ and $(\Phi'(a), ..., \Phi^{(\ell)}(a))$ are two bases of the direction of $\mathrm{Osc}_\ell\,\Phi(a)$. Consequently, the matrix $\mathscr{M}$ appearing in (2.30) satisfies $\mathscr{M}_{i,j}=0$ for $j>i$ and $\mathscr{M}_{i,i}\neq 0$. Hence,

$$\mathscr{B}_j^{(i)}(a)=0, \qquad \mathscr{B}_i^{(i)}(a)\neq 0 \qquad \text{for} \quad 1\leqslant i<j\leqslant n. \tag{2.31}$$

Relations (2.29), (2.31), and the symmetric ones obtained by exchanging the points a and b (which consists in taking the Chebyshev-Bernstein basis and the Chebyshev-Bézier points in reverse order) provide the expected result. ∎

Remark 2.11. (i) When Φ is a Chebyshev function, equalities (2.28) are also valid for $i=n$, but this is not true for quasi-Chebyshev functions. We can only write $\mathrm{aff}(\Pi_0, ..., \Pi_n)=\mathrm{Osc}_s\,\Phi(b)$, where s is the smallest integer such that $\Phi'(b), ..., \Phi^{(n-1)}(b), \Phi^{(s)}(b)$ are linearly independent. As a matter of fact, $\mathscr{B}_0$ vanishes exactly s times at b.

(ii) If the space $\mathscr{E}(\Phi)$ is additionally known to be a Chebyshev space on I (i.e., if the number of distinct zeros of any nonzero element of this space is bounded by n), then the properties stated in Theorem 2.10 prove that the Chebyshev-Bernstein basis is the *optimal shape preserving basis* of the restriction of the space $\mathscr{E}(\Phi)$ to $[a, b]$ (assuming that $a<b$), as introduced by J.-M. Carnicer and J.-M. Peña in [2]. It means that:

— it is *normalized* in the sense that $\sum_{i=0}^n \mathscr{B}_i=\mathbb{1}$,

— it is *totally positive*, i.e., for any $x_0<x_1<\cdots<x_n$ in $[a, b]$, the matrix $(\mathscr{B}_i(x_j))_{0\leqslant i,j\leqslant n}$ is totally positive (i.e., all its minors are nonnegative),

— all other basis of $\mathscr{E}(\Phi)$ assumed to be totally positive on $[a, b]$ is obtained by multiplication of $(\mathscr{B}_0, ..., \mathscr{B}_n)$ by a totally positive matrix.

For the proof we refer to [16, Theorem 3.2]. As for the importance of such bases in geometric design we refer to [2, 6, 8].

Corollary 2.3 has stated the linear independence of the n vectors $\Phi'(b), ..., \Phi^{(n-1)}(b), \Phi(b)-\Phi(a)$. Therefore, whatever the real numbers $\beta_0, ..., \beta_{n-1}, \alpha$, there exists a unique element $F \in \mathscr{E}(\Phi)$ such that $F^{(\ell)}(b)=\beta_\ell$, $\ell=0, ..., n-1$, and $F(a)=\alpha$. In particular, on account of Theorem 2.10, $\mathscr{B}_0$ is the unique element of $\mathscr{E}(\Phi)$ which vanishes n times at b and satisfies the additional condition $\mathscr{B}_0(a)=1$. We can more generally state the following result.

COROLLARY 2.12. *Given two distinct points $a, b \in I$ and the nonnegative integers i, j such that $i+j=n-1$, any Hermite interpolation problem*:

$$F^{(\ell)}(a)=\alpha_\ell, \quad \ell=0, ..., i, \qquad F^{(\ell)}(a)=\beta_\ell, \quad \ell=0, ..., j, \tag{2.32}$$

has a unique solution in the space $\mathscr{E}(\Phi)$.

Proof. As already observed, for $i=0$ or $j=0$, this follows from Corollary 2.3.

Consider an integer i, $1 \leqslant i \leqslant n-1$. According to (2.25), the Chebyshev-Bézier point Π_i is defined by: $\{\Pi_i\} := \mathrm{Osc}_i\, \Phi(a) \cap \mathrm{Osc}_{n-i}\, \Phi(b)$. Consequently, there exist unique real numbers $\lambda_{k,i}$ and $\mu_{\ell,i}$ such that

$$\Pi_{i+1}=\Phi(a)+\sum_{k=1}^{i+1} \lambda_{k,i}\, \Phi^{(k)}(a)=\Phi(b)+\sum_{\ell=1}^{n-i} \mu_{\ell,i}\, \Phi^{(\ell)}(b). \tag{2.33}$$

It results from (2.33) that

$$\lambda_{i,i}=\frac{\det(\Phi'(a), ..., \Phi^{(i-1)}(a), \Phi(b)-\Phi(a), \Phi'(b), ..., \Phi^{(n-i)}(b))}{\det(\Phi'(a), ..., \Phi^{(i)}(a), \Phi'(b), ..., \Phi^{(n-i)}(b))}. \tag{2.34}$$

Taking into account Theorem 2.10, by comparison of (2.30) with the first part of (2.33), we can see that $\lambda_{i,i}=1/\mathscr{B}_i^{(i)}(a)$. In particular $\lambda_{i,i} \neq 0$. This proves the linear independence of the n vectors $\Phi'(a), ..., \Phi^{(i-1)}(a)$, $\Phi(b)-\Phi(a)$, $\Phi'(b), ..., \Phi^{(n-i)}(b)$. The unicity of the solution of any Hermite interpolation problem such as (2.32) follows easily. ∎

In particular, for $i=1, ..., n-1$, the Chebyshev-Bernstein function $\mathscr{B}_i$ is the unique element of $\mathscr{E}(\Phi)$ which vanishes i times at a and $n-i$ times at b and satisfies the additional condition:

$$\mathscr{B}_i^{(i)}(a)=\frac{\det(\Phi'(a), ..., \Phi^{(i)}(a), \Phi'(b), ..., \Phi^{(n-i)}(b))}{\det(\Phi'(a), ..., \Phi^{(i-1)}(a), \Phi(b)-\Phi(a), \Phi'(b), ..., \Phi^{(n-i)}(b))}.$$

For the expressions of the Chebyshev-Bernstein basis which can be derived from these observations, we refer to [16].

2.4. *Splines Based on a Quasi-Chebyshev Function*

In this subsection we shall give a short outline on the contruction of splines based on the quasi-Chebyshev function Φ. If h is an affine map defined on the space $\mathcal{A}$, the Chebyshev-Bézier points (w.r. to (a, b)) of the $\mathcal{E}(\Phi)$-function $F = h \circ \Phi$ are defined as usual by $f(a^{n-i}, b^i)$, $i = 0, ..., n$, where $f = h \circ \varphi$ is the blossom of F. They are the images of the Chebyshev-Bézier points $\Pi_0, ..., \Pi_n$ of Φ under h. Through the following result, blossoms prove to be the relevant tool to characterize contact conditions between two $\mathcal{E}(\Phi)$-functions.

THEOREM 2.13. *Given any two $\mathcal{E}(\Phi)$-functions F and G, any point $a \in I$ and any integer r, $0 \leqslant r \leqslant n-1$, the following three properties are equivalent*:

(i) *F and G have a contact of order r at a, i.e., $F^{(i)}(a) = G^{(i)}(a)$ for $i = 0, ..., r$,*

(ii) *F and G have the same first $r+1$ Chebyshev-Bézier points w.r. to (a, b), where b is any point picked in $I \backslash \{a\}$,*

(iii) *the blossoms f and g of F and G take the same values at any n-tuple containing at least $(n-r)$ times the point a, i.e.,*

$$f(a^{n-r}, x_1, ..., x_r) = g(a^{n-r}, x_1, ..., x_r) \qquad \textit{for all} \quad x_1, ..., x_r \in I.$$

The proof is mainly based on (2.28) and on the definition of the blossoms f and g as $f = h \circ \varphi$ and $g = k \circ \varphi$, where h and k are the affine maps such that $F = h \circ \Phi$ and $G = k \circ \Phi$. It is similar to the case when Φ is a Chebyshev function which we refer to (see [15, 18]). However, let us mention the difference between the two cases. Here, the statement is no longer valid for $r = n$. If $r = n$, properties (ii) and (iii) are still equivalent, but they are no longer equivalent to (i). In order to make it clear, let us assume that $\Phi^{(n)}(a)$ is a linear combination of $\Phi'(a), ..., \Phi^{(n-1)}(a)$. Through Corollary 2.12, one can prove the existence of infinitely many affine maps $h: \mathcal{A} \to \mathcal{A}$ such that the corresponding $\mathcal{E}(\Phi)$-functions $F = h \circ \Phi$ satisfies $F^{(i)}(a) = \Phi^{(i)}(a)$ for $i = 0, ..., n-1$. All of them automatically satisfy $F^{(n)}(a) = \Phi^{(n)}(a)$, while the equality $f = \varphi$ holds only when h is the identity. Nevertheless, the important fact is the possibility of expressing contact conditions of order less than or equal to $n-1$ through blossoms. This allows the extension of the blossoming principle to splines associated with any given subdivision $t_0 < t_1 < \cdots < t_q < t_{q+1}$ contained in I, each knot t_i, $1 \leqslant i \leqslant q$, being allocated a multipicity $1 \leqslant m_i \leqslant n$. Let $\mathscr{S}$ denote the space of all functions S: $[t_0, t_1] \to \mathbb{R}$ which are C^{n-m_ℓ} at t_ℓ, $\ell = 1, ..., q$, and satisfy:

$$S(x) = F_k(x), \quad x \in [t_k, t_{k+1}], \qquad k = 0, ..., q,$$

where $F_k \in \mathscr{E}(\Phi)$. Using Corollary 2.12, it is easy to check that the space $\mathscr{S}$ is $(n+m+1)$-dimensional, where $m := \sum_{i=1}^{q} m_i$.

Now, introducing additional abscissae $t_{-n} \leqslant t_{-n+1} \leqslant \cdots \leqslant t_0$, $t_{q+1} \leqslant t_{q+2} \leqslant \cdots \leqslant t_{q+n}$ in the interval I, let us restrict ourselves to the simplest case of all multiplicities $m_1, ..., m_q$ equal to 1. Then, define the $(n+q+1)$ poles of S as

$$Q_j := f_k(t_{j+1}, ..., t_{j+n}), \qquad k \in \mathscr{J}_j, \; j = -n, ..., q,$$

where f_k denotes the blossom of F_k and

$$\mathscr{J}_j = \{\max(0, j), ..., \min(q, j+n)\}, \qquad j = -n, ..., q.$$

The fact that $Q_{-n}, ..., Q_q$ are well defined directly follows from Theorem 2.13, (iii). For any given $i = 0, ..., q$, and any $x \in [t_i, t_{i+1}]$, the properties of the blossoms then make it possible to calculate in n steps $S(x) = F_i(x)$ as a convex combination of the $n+1$ poles $Q_j = f_i(t_{j+1}, ..., t_{j+n})$, $j = i-n, ..., i$. This describes a de Boor type algorithm, leading as usual to a B-spline basis which satisfies all expected properties. For more details, we refer to [15, 18], everything working in the same way, including the case of higher multiplicities (provided that the multiplicities are greater than or equal to 1).

3. CONSTRUCTING QUASI-CHEBYSHEV FUNCTIONS FROM WEIGHT FUNCTIONS

A classical way to obtain extended Chebyshev spaces consists in defining them as kernels of differential operators associated with weight functions. Given positive functions $w_1, w_2, ..., w_n$: $I \to \mathbb{R}$, with $w_k \in C^{n+1-k}(I)$, we shall refer to them as *weight functions*. With such weight functions we associate the following differential operators defined on $C^n(I)$ by:

$$L_0 u := u, \quad L_k u := \left(\frac{1}{w_k} L_{k-1} u\right)', \qquad k = 1, ..., n. \tag{3.1}$$

It is well know that $\mathscr{U}_n := \operatorname{Ker}(L_n) = \{u \in C^n(I) \mid L_n u = 0\}$ is then an n-dimensional extended Chebyshev space on I, said to be associated with the weight functions $w_1, w_2, ..., w_n$ (see [21]). Starting from the same differential operators and from a given finite dimensional subspace $\mathscr{V}_0$ of $C^0(I)$, one can more generally consider

$$\mathscr{V}_n := \{u \in C^n(I) \mid L_n u \in \mathscr{V}_0\}. \tag{3.2}$$

Let us introduce the space $\mathscr{V}_1 := \{u \in C^1(I) \mid (u/w_n)' \in \mathscr{V}_0\}$. Using (3.1), we can as well define $\mathscr{V}_n$ as the space of all functions $u \in C^n(I)$ such that $L_{n-1} u \in \mathscr{V}_1$. From (3.1) it is easy to similarly check that

$$\mathscr{V}_n := \{u \in C^n(I) \mid L_{n-k} u \in \mathscr{V}_k\}, \tag{3.3}$$

where the $\mathscr{V}_k$'s are recursively defined by

$$\mathscr{V}_{k+1} := \left\{u \in C^{k+1}(I) \,\middle|\, \left(\frac{u}{w_{n-k}}\right)' \in \mathscr{V}_k\right\}, \qquad k = 0, ..., n-1. \tag{3.4}$$

Clearly $\dim \mathscr{V}_{k+1} = 1 + \dim \mathscr{V}_k$, so that

$$\dim \mathscr{V}_k = k + \dim \mathscr{V}_0, \qquad k = 0, ..., n. \tag{3.5}$$

Given any $v \in C^n(I)$, we say that a point $x \in I$ is a zero of multiplicity k of v, $0 \leqslant k \leqslant \cdots \leqslant n$, when

$$v^{(j)}(x) = 0 \quad \text{for } j = 0, ..., k-1, \qquad v^{(k)}(x) \neq 0,$$

and is a zero of multiplicity $n+1$ when

$$v^{(j)}(x) = 0 \qquad \text{for} \quad j = 0, ..., n.$$

For such a function v, we shall then denote by $Z_n^I(v)$ the total number of zeros of v in I, counted with this notion of multiplicity, this number being possibly equal to $+\infty$. For instance, if $v(x) := x^m$, where m is a positive integer, then $Z_k^{\mathbb{R}}(v) = k+1$ for all $k \leqslant m-1$, whereas $Z_k^{\mathbb{R}}(v) = m$ if $k \geqslant m$. For any $v \in C^0(I)$, $Z_0^I(v)$ is just the number of points at which v vanishes. Given a subspace $\mathscr{V}$ of $C^n(I)$, we shall denote by $Z_n^I(\mathscr{V})$ the upper bound of the quantities $Z_n^I(v)$, taken over all the nonzero elements v of $\mathscr{V}$. When $\mathscr{V}_0$ is 2-dimensional, it results from (3.5) that $\mathscr{V}_n$ is $(n+2)$-dimensional. The purpose of this section is to prove the following result.

THEOREM 3.1. *Assume the space $\mathscr{V}_0$ to be 2-dimensional, and consider $n+2$ functions $\Phi_1, ..., \Phi_{n+2} \in C^{n+1}(I)$ such that the first derivatives $\Phi_1', ..., \Phi_{n+2}'$ form a basis of $\mathscr{V}_n$. If $Z_0^I(\mathscr{V}_0) \leqslant 1$, then $\Phi := (\Phi_1, ..., \Phi_{n+2})^T$ is a quasi-Chebyshev function of order $n+2$ on I.*

Before proving this theorem, we shall illustrate it by a simple example. Let I be any interval and η a strictly monotonic continuous function on I. Then the space $\mathscr{V}_0 := \text{span}(\mathbf{1}, \eta)$ clearly satisfies $Z_0^I(\mathscr{V}_0) \leqslant 1$. Choosing $w_k := \mathbf{1}$ for all $k \geqslant 1$, we can see that, for all $k \geqslant 0$,

$$\mathscr{V}_k = \text{span}(1, ..., x^k, \eta_k(x)),$$

where η_k satisfies ${\eta_k}^{(k)} := \eta$. Therefore, for all $n \geqslant 2$, the function Φ defined on I by

$$\Phi(x) := (x, ..., x^{n-1}, \eta_{n-1}(x))^T,$$

is a quasi-Chebyshev function of order n on I. For instance, on any interval I we can take $\eta(x) := x^{2m+1}$ where m is a given nonnegative integer, which proves that $\Phi(x) := (x, ..., x^{n-1}, x^{n+2m})^T$ is a quasi-Chebyshev function of order n on $\mathbb{R}$. Observe that, unless $m = 0$ (in which case $\mathscr{E}(\Phi) = \mathscr{P}_n$), it is not a Chebyshev function on $\mathbb{R}$ since the last component of Φ vanishes $n + 2m$ times at 0.

In order to prove Theorem 3.1, we have to show that some intersections of affine flats consist of single points. We shall use the following technical lemmas, the proofs of which can be skipped at first reading.

LEMMA 3.2. *Given r linear subspaces $V_1, ..., V_r$ of $\mathbb{R}^q$, such that* $\dim(V_i) = q - \mu_i$ *with $\sum_{i=1}^r \mu_i = q$, for $i = 1, ..., r$, choose a basis $(v_1^i, ..., v_{q-\mu_i}^i)$ of V_i, and consider the $q \times (q - \mu_i)$ matrix $A_i := (v_1^i, ..., v_{q-\mu_i}^i)$. Let $a_1, ..., a_r$ be any r points of $\mathbb{R}^q$. Then, the affine flat $\bigcap_{i=1}^r (a_i + V_i)$ consists of a single point of $\mathbb{R}^q$ if and only if the following square matrix $\mathscr{A}$ of order $(r-1)\,q$ is regular*:

$$\mathscr{A} := \begin{pmatrix} A_1 & 0 & \cdots & 0 & A_r \\ 0 & A_2 & & 0 & A_r \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{r-1} & A_r \end{pmatrix}. \tag{3.6}$$

Proof. Considering the $r-1$ vectors of $\mathbb{R}^q$ defined by $X_i := a_i - a_r$, $i = 1, ..., r-1$, the intersection $\bigcap_{i=1}^r (a_i + V_i)$ consists of a single point iff there exist unique vectors $x_1, ..., x_r$, with $x_i \in V_i$, $i = 1, ..., r$, satisfying

$$X_i = x_i + x_r \quad i = 1, ..., r-1,$$

or equivalently, unique real numbers λ_j^i, $i=1, ..., r$, $j=1, ..., q-\mu_i$, such that

$$X_i = \sum_{j=1}^{q-\mu_i} \lambda_j^i v_j^i + \sum_{j=1}^{q-\mu_r} \lambda_j^r v_j^r .$$

This is a linear system of order $(r-1)\,q$ the matrix of which is $\mathscr{A}$, whence the result. ▮

Lemma 3.3. *The data are the same as in Lemma 3.2. Given $k=1, ..., r$, for each subset L_k of the set $\{1, ..., q-\mu_k\}$, with $|L_k| = p$, let us denote by $A_k(L_k)$ the $q \times p$ matrix defined by*

$$A_k(L_k) := (v_{\ell_1}^k, ..., v_{\ell_p}^k),$$

where $\ell_1 < \ell_2 < \cdots < \ell_p$ are the elements of L_k. Let us also set $\overline{L_k} := \{1, ..., q-\mu_k\} \backslash L_k$ and $L_k^* := \ell_1 + \cdots + \ell_p$. Then, the determinant of the square matrix $\mathscr{A}$ defined in (3.6) *satisfies*:

$$\det \mathscr{A} = \pm \sum_{\substack{L_k \subset \{1, ..., q-\mu_k\} \\ |L_k| = \mu_{k+1} + \cdots + \mu_r \\ 1 \leqslant k \leqslant r}} \prod_{k=1}^{r-1} (-1)^{L_k^*} \det(A_k(L_k), A_{k+1}(\overline{L_{k+1}})). \tag{3.7}$$

Proof. Let us start by a comment on formula (3.7). For $k=1$, the two conditions $L_1 \subset \{1, ..., q-\mu_1\}$ and $|L_1| = \mu_2 + \cdots + \mu_r$ are satisfied only for $L_1 = \{1, ..., q-\mu_1\}$, which means that $A_1(L_1) = A_1$. Similarly, for $k=r$, the only possibility is $L_r = \varnothing$, and thus, $A_r(\overline{L_r}) = A_r$.

In order to prove equality (3.7), the first step consists in observing that $\det \mathscr{A} = \pm \det \mathscr{B}$, where

$$\mathscr{B} := \begin{pmatrix} A_1 & 0 & \cdots & 0 & A_{r-1} & 0 \\ 0 & A_2 & \cdots & 0 & A_{r-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{r-2} & A_{r-1} & 0 \\ 0 & 0 & \cdots & 0 & A_{r-1} & A_r . \end{pmatrix}. \tag{3.8}$$

Let us denote by $\mathscr{B}_1^1, ..., \mathscr{B}_{q-\mu_1}^1, \mathscr{B}_1^2, ..., \mathscr{B}_{q-\mu_2}^2, ... \in \mathbb{R}^{(r-1)\,q}$ the columns of matrix $\mathscr{B}$. For $1 \leqslant j \leqslant q-\mu_{r-1}$, $\mathscr{B}_j^{r-1\,T} = (v_j^{r-1\,T}, ..., v_j^{r-1\,T})$, so that we can write

$$\mathscr{B}_j^{r-1} = \mathscr{C}_j + \mathscr{D}_j , \qquad j=1, ..., q-\mu_{r-1}, \tag{3.9}$$

where

$$\mathscr{C}_j^T := (0_{\mathbb{R}^q}{}^T, ..., 0_{\mathbb{R}^q}{}^T, v_j^{r-1\,T}), \qquad \mathscr{D}_j^T := (v_j^{r-1\,T}, ..., v_j^{r-1\,T}, 0_{\mathbb{R}^q}{}^T).$$

The multilinearity of the determinant with respect to its columns yields

$$\det \mathscr{B} = \sum_{L \subset \{1, ..., q-\mu_{r-1}\}} \det \mathscr{B}(L), \tag{3.10}$$

where

$$\mathscr{B}(L) := (..., \mathscr{B}^{r-2}_{q-\mu_{r-2}}, \mathscr{B}^{r-1}_1(L), ..., \mathscr{B}^{r-1}_{q-\mu_{r-1}}(L), \mathscr{B}^r_1, ...)$$

is obtained from $\mathscr{B}$ by replacing the columns $\mathscr{B}^{r-1}_j$ by

$$\mathscr{B}^{r-1}_j(L) := \mathscr{C}_j \quad \text{if } j \in L, \qquad \mathscr{B}^{r-1}_j(L) := \mathscr{D}_j \quad \text{if } j \notin L, \quad j = 1, ..., q-\mu_{r-1}. \tag{3.11}$$

Picking any sequence $\ell_1 < \cdots < \ell_p$ in the set $\{1, ..., q-\mu_{r-1}\}$, let us calculate $\det \mathscr{B}(L)$ for $L := \{\ell_1, ..., \ell_p\}$. Change the order of the columns $\mathscr{B}^{r-1}_j(L)$ so that $\mathscr{C}_{\ell_1}, ..., \mathscr{C}_{\ell_p}$ come last. Using the notations introduced in the statement of the lemma, we obtain

$$\det \mathscr{B}(L) = (-1)^{p(q-\mu_{r-1}) + 1/2(p-1)p + L^*} \det \tilde{\mathscr{B}}(L), \tag{3.12}$$

where $\tilde{\mathscr{B}}(L)$ is the following matrix:

$$\tilde{\mathscr{B}}(L) := \begin{pmatrix} A_1 & \cdots & 0 & A_{r-1}(\bar{L}) & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & A_{r-2} & A_{r-1}(\bar{L}) & 0 & 0 \\ 0 & \cdots & 0 & 0 & A_{r-1}(L) & A_r \end{pmatrix}. \tag{3.13}$$

Calculating $\det \tilde{\mathscr{B}}(L)$ by block, we can check that it is equal to zero (and thus, due to (3.12), so is $\det \mathscr{B}(L)$), unless $p = \mu_r$. Equality (3.10) therefore reduces to

$$\det \mathscr{B} = \sum_{\substack{L \subset \{1, ..., q-\mu_{r-1}\} \\ |L| = \mu_r}} \det \mathscr{B}(L).$$

Now, using (3.12) and (3.13), this finally yields

$$\det \mathscr{B} = (-1)^{\mu_r(q-\mu_{r-1})+1/2(\mu_r-1)\mu_r} \sum_L (-1)^{L^*} \det(A_{r-1}(L), A_r)$$
$$\times \begin{vmatrix} A_1 & \cdots & 0 & A_{r-1}(\bar{L}) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_{r-2} & A_{r-1}(L) \end{vmatrix},$$

the sum in (3.14) being taken over all subsets L of $\{1, ..., q-\mu_{r-1}\}$ such that $|L| = \mu_r$. The matrices appearing in the right hand side of (3.14) have exactly the same structure as $\mathscr{A}$. We can thus repeat the procedure, which eventually leads to (3.7). ∎

Proof of Theorem 3.1. In order to prove that Φ is a quasi-Chebyshev function of order $n+2$ on I, we actually have to check that, for any integer $r \geqslant 2$, any given points $\tau_1 < \cdots < \tau_r$ in I, and any given positive integers $\mu_1, ..., \mu_r$ such that $\sum_{i=1}^r \mu_i = n+2$, the intersection

$$\bigcap_{i=1}^{r} \operatorname{Osc}_{n+2-\mu_i} \Phi(\tau_i)$$

consists of a single point. The situation is thus the same as that investigated in Lemmas 3.2 and 3.3, with $q := n+2$ and, for $i = 1, ..., r$,

$$a_i := \Phi(\tau_i), \qquad V_i := \operatorname{span}(\Phi'(\tau_i), ..., \Phi^{(n+2-\mu_i)}(\tau_i)).$$

Observe that, for any $u \in C^n(I)$ and any $x \in I$,

$$(L_0 u(x), ..., L_n u(x))^T = \mathscr{T}(x) \cdot (u(x), ..., u^{(n)}(x))^T, \tag{3.15}$$

where $\mathscr{T}(x)$ is a lower triangular matrix with diagonal elements $(1/\prod_{k=1}^i w_k(x))_{i=0, ..., n}$. Consequently, as a basis of V_i we can choose

$$v_j^i := L_{j-1} \Phi'(\tau_i), \qquad j = 1, ..., n+2-\mu_i.$$

According to Lemma 3.2, we have to check that the corresponding matrix $\mathscr{A}$ defined by (3.6) is regular. Due to equality (3.7) it is sufficient to prove that, for a given $k = 1, ..., r$, all the corresponding quantities $(-1)^{L_k^*}$ $\det(A_k(L_k), A_{k+1}(\overline{L_{k+1}}))$ have the same strict sign, whatever L_k, $\overline{L_{k+1}}$ such that $L_k \subset \{1, ..., q-\mu_k\}$, $|L_k| = \mu_{k+1} + \cdots + \mu_r$, $\overline{L_{k+1}} \subset \{1, ..., q-\mu_{k+1}\}$, $|\overline{L_{k+1}}| = \mu_1 + \cdots + \mu_k$. As a matter of fact, this will result by applying Proposition 3.4 below to $\Psi := \Phi'$. ∎

PROPOSITION 3.4. *Assume* $\mathscr{V}_0$ *to be 2-dimensional and to satisfy* $Z_0^I(\mathscr{V}_0) \leqslant 1$. *Let* $\Psi := (\Psi_1, ..., \Psi_{n+2})^T$, *where* $(\Psi_1, ..., \Psi_{n+2})$ *is a basis of* $\mathscr{V}_n$. *Then, for any integer* μ, $0 \leqslant \mu \leqslant n$, *the quantity*

$$(-1)^{\ell_0 + \cdots + \ell_\mu} \det[L_{\ell_0}\Psi(a), ..., L_{\ell_\mu}\Psi(a), L_{m_0}\Psi(b), ..., L_{m_{n-\mu}}\Psi(b)] \tag{3.16}$$

has the same strict sign for any sequences of integers $0 \leqslant \ell_0 < \ell_1 < \cdots < \ell_\mu \leqslant n$, $0 \leqslant m_0 < m_1 < \cdots < m_{n-\mu} \leqslant n$, *and any* $a, b \in I$, $a < b$.

Let us first state two elementary lemmas which will prove to be useful throughout the proof of Proposition 3.4. Starting from any function $v \in C^n(I)$, L_k being a differential operator of order k, $L_k v \in C^{n-k}(I)$ and thus $Z_{n-k}^I(L_k v)$ is well defined.

LEMMA 3.5. *For any interval* $J \subset I$, *any function* $v \in C^n(I)$ *and any integer* k, $0 \leqslant k \leqslant n-1$,

$$Z_{n-k-1}^J(L_{k+1}v) \geqslant Z_{n-k}^J(L_k v) - 1. \tag{3.17}$$

Furthermore, suppose that there is equality in (3.17) *and that* $J = [a, b]$. *Then*

$$L_k v(a) \neq 0 \Rightarrow L_k v(a)\, L_{k+1}v(a) < 0, \tag{3.18}$$

$$L_k v(b) \neq 0 \Rightarrow L_k v(b)\, L_{k+1}v(b) > 0. \tag{3.19}$$

Proof. Since $L_{k+1}v = (\frac{1}{w_{k+1}} L_k v)'$ and $Z_{n-k}^J(L_k v) = Z_{n-k}^J(\frac{1}{w_{k+1}} L_k v)$, (3.17) follows from Rolle's theorem.

Suppose that $J = [a, b]$ and that there is equality in (3.17). It follows that $L_k v$ vanishes at least once in J. If $L_k v(a) \neq 0$, let $\alpha \in]a, b]$ be the smallest point in J at which $L_k v$ vanishes. Then, $L_{k+1}v$ does not vanish on $[a, \alpha[$ (for instance, in case $\alpha < b$, this results by applying inequality (3.17) on $[\alpha, b]$), which proves $\frac{1}{w_{k+1}} L_k v$ to be monotonic on $[a, \alpha]$. Consequently, due to $L_k(a) \neq 0$, and $L_k(\alpha) = 0$, we can state that

$$\frac{1}{w_{k+1}} L_k v(a) \left(\frac{1}{w_{k+1}} L_k v\right)' < 0,$$

i.e., $L_k v(a)\, L_{k+1}v(a) < 0$. A similar argument leads to (3.19). ∎

LEMMA 3.6. *Assume that* $Z_0^I(\mathscr{V}_0) \leqslant M$. *Then, for any integer* $k \leqslant n$, $Z_k^I(\mathscr{V}_k) \leqslant k + M$. *Moreover, given* $v \in \mathscr{V}_n$ *and* $J \subset I$, *if* $Z_{n-\ell}^J(L_\ell v) = n - \ell + M$ *for a given integer* ℓ, $0 \leqslant \ell \leqslant n$, *then* $Z_{n-k}^J(L_k v) = n - k + M$ *for any integer* k, $n \geqslant k \geqslant \ell$.

Proof. This is an immediate consequence of (3.17). ∎

Proof of Proposition 3.4. According to Lemma 3.6, $Z^I_k(\mathscr{V}_k) \leqslant k+1$ for all $k \leqslant n$. So, if $v \in \mathscr{V}_n$ satisfies $Z^I_n(v) = n+2$, then $v = 0$. Therefore, given any integer r, $2 \leqslant r \leqslant n+2$, any distinct points $a_1, ..., a_r \in I$, and any positive integers $\mu_1, ..., \mu_r$ such that $\sum_{i=1}^r \mu_i = n+2$, we must have

$$\det[\Psi(a_1), \Psi'(a_1), ..., \Psi^{(\mu_1-1)}(a_1), \Psi(a_2), ..., \Psi(a_r), ..., \Psi^{(\mu_r-1)}(a_r)] \neq 0.$$

In particular, by continuity, it follows that, for a fixed integer μ, $0 \leqslant \mu \leqslant n$, the determinant

$$\det[\Psi(a), \Psi'(a), ..., \Psi^{(\mu)}(a), \Psi(b), \Psi'(b), ..., \Psi^{(n-\mu)}(b)] \tag{3.20}$$

keeps the same strict sign for any $a, b \in I$, $a < b$. Hence, so does

$$\Delta(a, b) := \det[L_0 \Psi(a), L_1 \Psi(a), ..., L_\mu \Psi(a), L_0 \Psi(b), L_1 \Psi(b), ..., L_{n-\mu} \Psi(b)], \tag{3.21}$$

since, by (3.15), $\Delta(a, b)$ is obtained by dividing (3.20) by $\prod_{i=1}^{\mu} w_i(a)^{\mu-i+1}$ $\prod_{j=1}^{n-\mu} w_j\,(b)^{n-\mu-j+1}$ which is positive.

Now, let us set:

$$u(x) := \det[L_0 \Psi(a), ..., L_{\mu-1} \Psi(a), \Psi(x), L_0 \Psi(b), ..., L_{n-\mu} \Psi(b)]. \tag{3.22}$$

Then, u clearly belongs to $\mathscr{V}_n$ and $L_j u(x)$ is obtained by replacing $\Psi(x)$ by $L_j \Psi(x)$ in the right hand side of (3.22). Accordingly, we have:

$$\begin{aligned} &L_0 u(a) = \cdots = L_{\mu-1} u(a) = 0, \\ &L_0 u(b) = \cdots = L_{n-\mu} u(b) = 0, \\ &L_\mu u(a) = \Delta(a, b) \neq 0. \end{aligned} \tag{3.23}$$

Therefore, from (3.15) we can derive that $Z^{[a,b]}_n(u) \geqslant n+1$. The inequality $Z^I_n(\mathscr{V}_n) \leqslant n+1$ shows that in fact $Z^{[a,b]}_n(u) = n+1$. On account of Lemma 3.6, this implies that $Z^{[a,b]}_{n-j}(L_j u) = n-j+1$ for all $j = 0, ..., n$. We thus have in particular

$$L_\mu u(a) \neq 0, \qquad Z^{[a,b]}_{n-k-1}(L_{k+1} u) = Z^{[a,b]}_{n-k}(L_k u) - 1, \qquad \mu \leqslant k \leqslant j-1,$$

for all $j = \mu+1, ..., n$. Applying (3.18) recursively, it follows that

$$\begin{aligned} (-1)^{j-\mu} L_\mu u(a)\, L_j u(a) = (-1)^{j-\mu} \Delta(a, b)\, L_j u(a) > 0 \\ \text{for all} \quad j = \mu, ..., n, \end{aligned} \tag{3.24}$$

which can also be written as follows:

$$(-1)^{1+\cdots+\mu} \Delta(a,b)\,(-1)^{1+\cdots+(\mu-1)+j} L_j u(a) > 0 \qquad \text{for all} \quad j=\mu, ..., n.$$

We actually have proved that, whatever $a<b$, all the quantities (3.16) have the same strict sign as $(-1)^{1+\cdots+\mu} \Delta(a,b)$ provided that $\ell_i = i$ for $0 \leqslant i \leqslant \mu-1$ and $m_i = i$ for $0 \leqslant i \leqslant n-\mu$. Let us now fix an integer ℓ_μ such that $\mu+1 \leqslant \ell_\mu \leqslant n$, and define:

$$v(x) := \det[L_0 \Psi(a), ..., L_{\mu-2} \Psi(a), \Psi(x), L_{\ell_\mu} \Psi(a), L_0 \Psi(b), ..., L_{n-\mu} \Psi(b)]. \tag{3.25}$$

Again, this function v belongs to $\mathscr{V}_n$ and it clearly satisfies:

$$\begin{aligned} L_0 v(a) = \cdots = L_{\mu-2} v(a) = 0, \\ L_0 v(b) = \cdots = L_{n-\mu} v(b) = 0, \\ L_{\ell_\mu} v(a) = 0, \\ L_{\mu-1} v(a) = L_{\ell_\mu} u(a). \end{aligned} \tag{3.26}$$

Accordingly, due to (3.24), $L_{\mu-1} v(a) \neq 0$. On the other hand, (3.26) also shows that $Z_n^{[a,b]}(v) \geqslant n$. Applying (3.17) then proves that

$$Z_{n-j}^{[a,b]}(L_j v) \geqslant n-j \qquad \text{for all} \quad j=0, ..., n. \tag{3.27}$$

From $L_{\mu-1} v(a) \neq 0$ we can deduce that $Z_{n-\mu+1}^{[a,b]}(L_{\mu-1} v) = Z_{n-\mu+1}^{]a,b]}(L_{\mu-1} v)$. Hence, applying (3.17) again, but now on $]a,b]$, we obtain:

$$n-j+1 \geqslant Z_{n-j}^I(\mathscr{V}_j) \geqslant Z_{n-j}^{]a,b]}(L_j v) \geqslant n-j, \qquad \mu-1 \leqslant j \leqslant n. \tag{3.28}$$

Let us assume that, for some integer j, $\mu-1 \leqslant j \leqslant \ell_\mu - 1$, $Z_{n-j}^{]a,b]}(L_j v) = n-j+1$. According to Lemma 3.6, we would have:

$$Z_{n-k}^{]a,b]}(L_k v) = n-k+1 \qquad \text{for all} \quad k=j, ..., n. \tag{3.29}$$

Thus, in particular, due to $L_{\ell_\mu} v(a) = 0$, (3.29) would imply

$$Z_{n-\ell_\mu}^{[a,b]}(L_{\ell_\mu} v) \geqslant 1 + Z_{n-\ell_\mu}^{]a,b]}(L_{\ell_\mu} v) = n - \ell_\mu + 2,$$

which would contradict $Z_{n-\ell_\mu}^I(\mathscr{V}_{n-\ell_\mu}) \leqslant n-\ell_\mu+1$. Taking (3.28) into consideration, we finally have proved that

$$Z_{n-j}^{]a,b]}(L_j v) = n-j \qquad \text{for all} \quad j=\mu-1, ..., \ell_\mu - 1. \tag{3.30}$$

On the other hand, suppose that $L_{\ell_\mu - 1}v(a) = 0 = L_{\ell_\mu}v(a)$. The point a would then be a zero of $L_{\ell_\mu - 1}v$ of multiplicity $\geqslant 2$. On account of (3.30), we would thus have

$$Z^{[a,b]}_{n-\ell_\mu+1} \geqslant 2 + Z^{]a,b]}_{n-\ell_\mu+1} = n - \ell_\mu + 3,$$

which would contradict $Z^{[a,b]}_{n-\ell_\mu+1} \leqslant n - \ell_\mu + 2$. Therefore $L_{\ell_\mu - 1}v(a) \neq 0$. Consequently, (3.30) leads to:

$$Z^{[a,b]}_{n-\ell_\mu+1}(L_{\ell_\mu-1}v) = Z^{]a,b]}_{n-\ell_\mu+1}(L_{\ell_\mu-1}v) = n - \ell_\mu + 1. \tag{3.31}$$

From (3.31) and (3.17) we can derive that $Z^{[a,b]}_{n-j}(L_j v) \leqslant n - j$ for $j = \mu - 1, ..., \ell_\mu - 1$. By comparison with (3.30), this eventually proves that

$$Z^{[a,b]}_{n-j}(L_j v) = n - j, \qquad L_j v(a) \neq 0 \quad \text{for } j = \mu - 1, ..., \ell_\mu - 1.$$

All inequalities (3.17) on $[a, b]$, corresponding to $k = \mu - 1, ..., \ell_\mu - 2$, are therefore equalities. Starting from $L_{\mu-1}v(a) \neq 0$ and using (3.18), we can conclude that

$$(-1)^{j-\mu+1}\, L_{\mu-1}v(a)\, L_j v(a) > 0 \qquad \text{for } j = \mu - 1, ..., \ell_\mu - 1.$$

Writing equality (3.24) for $j = \ell_\mu$, and taking into account the fact that $L_{\mu-1}v(a) = L_{\ell_\mu}u(a)$, the latter equality leads to

$$(-1)^{j-\mu+1}\,(-1)^{\ell_\mu - \mu}\, \Delta(a, b)\, L_j v(a) > 0 \qquad \text{for} \quad j = \mu - 1, ..., \ell_\mu - 1, \tag{3.32}$$

that is

$$(-1)^{1+\cdots+\mu}\, \Delta(a, b)\, (-1)^{1+\cdots+(\mu-2)+j+\ell_\mu} L_j v(a) > 0,$$
$$j = \mu - 1, ..., \ell_\mu - 1.$$

This means that all the quantities (3.16) corresponding to $\ell_i = i$ for $j = 0, ..., \mu - 2$, and $m_i = i$ for $i = 0, ..., n - \mu$ have the same strict sign as $(-1)^{1+\cdots+\mu}\,\Delta(a, b)$. The next step will consist in fixing a second integer $\ell_{\mu-1}$ such that $\mu \leqslant \ell_{\mu-1} < \ell_\mu \leqslant n$ and in considering the function $w \in \mathscr{V}_n$ defined by

$$w(x) := \det[L_0\Psi(a), ..., L_{\mu-3}\Psi(a), \Psi(x),$$
$$L_{\ell_{\mu-1}}\Psi(a), L_{\ell_\mu}\Psi(a), L_0\Psi(b), ..., L_{n-\mu}\Psi(b)].$$

This function clearly satisfies $L_{\mu-2}w(a)=L_{\ell_{\mu_1}}v(a)$. Similar arguments will make it possible to prove that

$$Z_{n-j}^{[a,b]}(L_j w)=n-j-1, \quad L_j w(a)\neq 0 \qquad \text{for} \quad j=\mu-2, ..., \ell_{\mu-1}-1,$$

thus leading to

$$(-1)^{j-\mu+2}\, L_{\mu-2}w(a)\; L_j w(a)>0, \qquad j=\mu-2, ..., \ell_{\mu-1}-1$$

by application of (3.18). Due to (3.32) this will show that, for $a<b$, all the quantities (3.16) have the same strict sign as $(-1)^{1+\cdots+\mu}\,\Delta(a,b)$ provided that $\ell_i=i$ for $i\leqslant\mu-3$ and $m_i=i$ for all $i\leqslant n-\mu$. Continuing in this way proves the expected result provided that $m_i=i$ for $i=0, ..., n-\mu$. Applying the same procedure at b rather than a yields the final result. ▮

Remarks 3.7. (i) Select a basis (α, β) in the space $\mathscr{V}_0$ and a point $a\in I$. Then we can choose $\Phi=(\Phi_1, ..., \Phi_{n+2})^T$ as follows:

$$\begin{aligned}
\Phi_1(x) &:= \int_a^x w_1(t_1)\,dt_1,\\
\Phi_2(x) &:= \int_a^x w_1(t_1)\int_a^{t_1} w_2(t_2)\,dt_2 dt_1,\\
&\;\vdots\\
\Phi_n(x) &:= \int_a^x w_1(t_1)\int_a^{t_1} w_2(t_2)\int_a^{t_2}\cdots\int_a^{t_{n-1}} w_n(t_n)\,dt_n\cdots dt_2 dt_1, \qquad (3.33)\\
\Phi_{n+1}(x) &:= \int_a^x w_1(t_1)\int_a^{t_1} w_2(t_2)\int_a^{t_2}\cdots\\
&\qquad\times\int_a^{t_{n-1}} w_n(t_n)\int_a^{t_n}\alpha(t)\,dt\,dt_n\cdots dt_2\,dt_1,\\
\Phi_{n+2}(x) &:= \int_a^x w_1(t_1)\int_a^{t_1} w_2(t_2)\int_a^{t_2}\cdots\\
&\qquad\times\int_a^{t_{n-1}} w_n(t_n)\int_a^{t_n}\beta(t)\,dt\,dt_n\cdots dt_2\,dt_1.
\end{aligned}$$

With $\Phi_0=\mathbf{1}$, the basis $(\Phi_0, ..., \Phi_{n+2})$ generalises the canonical systems classically introduced in extended Chebyshev spaces (see [21]). If everything is assumed to be C^∞ on I, then, for $i\geqslant 1$,

$$\det(L_0\Phi', ..., L_n\Phi', \Phi^{(n+1+i)})=w_1\cdots w_n\begin{vmatrix}\alpha & \alpha^{(i)}\\ \beta & \beta^{(i)}\end{vmatrix}.$$

Condition (H_{n+2}) of Section 2.3 is therefore satisfied iff, for any $x \in I$, there exists an integer $i \geqslant 1$ such that $\alpha(x)\,\beta^{(i)}(x) - \alpha^{(i)}(x)\,\beta(x) \neq 0$.

(ii) Under the assumptions of Theorem 3.1, the associated space $\mathscr{E}(\Phi)$ satisfies $\mathscr{E}(\Phi) = \{u \in C^{n+1}(I) \mid L_n u' \in \mathscr{V}_0\}$. The condition $Z_0^I(\mathscr{V}_0) \leqslant 1$ means that $\mathscr{V}_0$ is a Chebyshev space on I. According to Lemma 3.6, it implies in particular that $\mathscr{V}_n$ is also a Chebyshev space on I and therefore so is $\mathscr{E}(\Phi)$. Consequently, on account of Remark 2.11, the Chebyshev-Bernstein basis w.r. to any $a, b \in I$, $a < b$, is the optimal shape preserving basis of $\mathscr{E}(\Phi)$ restricted to $[a, b]$.

(iii) Suppose now that $\mathscr{V}_0 \subset C^2(I)$ and that it is a 2-dimensional extended Chebyshev space on I. Then, if I is a closed bounded interval, there exist weight functions $w_{n+1} \in C^2(I)$, $w_{n+2} \in C^1(I)$ such that $\mathscr{V}_0 = \operatorname{Ker} M_2$, where, for all $u \in C^2(I)$,

$$M_1 u := \left(\frac{1}{w_{n+1}}\, u\right)', \qquad M_2 u := \left(\frac{1}{w_{n+2}}\, M_1 u\right)'.$$

Assuming the weight functions $w_1, ..., w_n$ to satisfy $w_k \in C^{n+3-k}(I)$, the differential operators $L_0, ..., L_n$ can now be defined on $C^{n+2}(I)$, and the space $\mathscr{V}_n$ is the extended Chebyshev space associated with the weight functions $w_1, ..., w_{n+2}$. The function Φ appearing in Theorem 3.1 is therefore a Chebyshev function of order $n+2$ on I. The result remains true even if I is not closed and bounded, since Φ is proved to be a Chebyshev function on any closed bounded interval contained in I.

4. EXAMPLES

In this section we use the method of Section 3 to construct examples of quasi-Chebyshev functions comprising polynomials which satisfy condition (H_n) of Section 2.3 and hence enjoy the properties derived in Sections 2.3 and 2.4. As discussed in Section 1, these examples include those introduced in [3, 10] for tension methods for shape-preserving interpolation.

Take $I := [0, 1]$, with $\mathscr{V}_0 := \operatorname{span}\ (\alpha, \beta)$, where $\alpha(x) := x^{m_1}$, $\beta(x) := (1-x)^{m_2}$ (m_1 and m_2 being any two positive integers), and $w_k = \mathbf{1}$, $k = 1, ..., n-2$. Associated with these data, consider the spaces $\mathscr{V}_k$, $k = 0, ..., n-2$, defined as in (3.4). We clearly have:

$$\mathscr{V}_k = \operatorname{span}(1, x, ..., x^{k-1}, x^{k+m_1}, (1-x)^{k+m_2}), \qquad k = 0, ..., n-2. \tag{4.1}$$

Let us introduce the function $\Phi: I \to \mathbb{R}^n$ defined by:

$$\Phi(x) := (\Phi_1(x), ..., \Phi_n(x))^T$$
$$:= (x, ..., x^{n-2}, x^{n-1+m_1}, (1-x)^{n-1+m_2})^T, \qquad x \in [0, 1]. \qquad (4.2)$$

Since $(\Phi_1', ..., \Phi_n')$ is a basis of the space $\mathscr{V}_{n-2}$, according to Theorem 3.1, function Φ will be proved to be a quasi-Chebyshev function of order n on I after checking that $Z_0^I(\mathscr{V}_0) \leqslant 1$. Let us observe that some geometric properties of the space $\mathscr{E}(\Phi)$ were studied in [5] in the particular case $n=3$.

Now, consider a nonzero element $w \in \mathscr{V}_0$. If w vanishes at 0, $w = \lambda \alpha$ for some $\lambda \neq 0$, and thus does not vanish elsewhere on I. A similar argument works at 1. On the other hand, from the fact that both $\alpha(x)$ and the determinant

$$\begin{vmatrix} \alpha(x) & \alpha'(x) \\ \beta(x) & \beta'(x) \end{vmatrix} = x^{m_1-1}\,(1-x)^{m_2-1}\left[(m_1-m_2)\,x - m_1\right]$$

do not vanish on]0, 1[, we can deduce than $\mathscr{V}_0$ is an extended Chebyshev space on]0, 1[. Therefore, w has at most one zero in this interval. We thus have checked that $Z_0^I(\mathscr{V}_0) = 1$.

Furthermore, the fact that $\mathscr{V}_0$ is an extended Chebyshev space on]0, 1[also implies that each $\mathscr{V}_k$ is a $(k+2)$-dimensional extended Chebyshev space on]0, 1[(see Remark 3.7, (iii)). Therefore, for all $x \in\,]0, 1[$, the n derivative vectors $\Phi'(x), ..., \Phi^{(n)}(x)$ are linearly independent. Moreover, it is easy to check that

$$\alpha(0)\,\beta^{(m_1)}(0) - \alpha^{(m_1)}(0)\,\beta(0) = -m_1!,$$
$$\alpha(1)\,\beta^{(m_2)}(1) - \alpha^{(m_2)}(1)\,\beta(1) = (-1)^{m_2}\,m_2!.$$

According to Remark 3.7, (i), this proves the linear independence of the n vectors $\Phi'(0), ..., \Phi^{(n-1)}(0), \Phi^{(n-1+m_1)}(0)$ on the one hand, and of $\Phi'(1), ..., \Phi^{(n-1)}(1), \Phi^{(n-1+m_2)}(1)$ on the other hand. Therefore the assumption (H_n) is satisfied (this could also have been derived from α and β being analytic on I). Function Φ thus meets all the requirements of the previous sections, making it possible to develop the subblossoming principle, and consequently the de Casteljau Chebyshev algorithm w.r. to any distinct points $a, b \in I$. Nevertheless, a *dimension elevation* process can be developed exactly as in the case of Chebyshev functions (see [17, 19]). Therefore, instead of working in the space $\mathscr{E}(\Phi)$, it may be more efficient to take advantage of the fact that $\mathscr{E}(\Phi) \subset \mathscr{P}_{n-1+\max(m_1, m_2)}$ so as to use the classical algorithms for polynomials. This will be illustrated in the particular case

$m_1 = m_2$ which will shall now focus on. So, in the following, given two integers n, m, with $n \geqslant 2$ and $m \geqslant 0$, Φ will denote the quasi-Chebyshev fonction of order n on $I = [0, 1]$ defined by:

$$\begin{aligned}\Phi(x) &:= (\Phi_1(x), ..., \Phi_n(x))^T \\ &:= (x, ..., x^{n-2}, x^{n+m}, (1-x)^{n+m})^T, \qquad x \in [0, 1]. \end{aligned} \tag{4.3}$$

We shall denote its blossom as usual by $\varphi = (\varphi_1, ..., \varphi_n)^T \colon I^n \to \mathbb{R}^n$. Let $\Pi_0, ..., \Pi_n$ be its Chebyshev-Bézier points with respect to $(0, 1)$, i.e.,

$$\Pi_i := \varphi(0^{n-i}, 1^i), \qquad i = 0, ..., n. \tag{4.4}$$

Clearly, this function Φ can also be viewed as a polynomial function of degree $n+m$, that is, as an element of $\mathscr{P}^n_{n+m}$. We can thus consider the $(n+m+1)$ Bézier points of Φ (w.r. to $(0, 1)$). The following proposition states how to construct them from its Chebyshev-Bézier points $\Pi_0, ..., \Pi_n$.

PROPOSITION 4.1. *Let $\hat{\Pi}_0, ..., \hat{\Pi}_{n+m}$ be the $(n+m+1)$ Bézier points of Φ viewed as a polynomial function of degree $n+m$. Then $\hat{\Pi}_0 = \Pi_0$ and $\hat{\Pi}_{n+m} = \Pi_n$, while the points $\hat{\Pi}_1, ..., \hat{\Pi}_{n+m-1}$ are obtained by applying the classical degree elevation process from degree $n-2$ to degree $n+m-2$, starting from the Chebyshev-Bézier points $\Pi_1, ..., \Pi_{n-1}$.*

Proof. Let us denote by $\hat{\varphi} = (\hat{\varphi}_1, ..., \hat{\varphi}_n)^T \colon I^{n+m} \to \mathbb{R}^n$ the ordinary blossom of Φ considered as an element of $\mathscr{P}_{n+m}^{\;n}$. Then the Bézier points $\hat{\Pi}_0, ..., \hat{\Pi}_{n+m}$ are defined by:

$$\hat{\Pi}_i := \hat{\varphi}(0^{n+m-i}, 1^i), \qquad i = 0, ..., n+m. \tag{4.5}$$

Of course, $\hat{\Pi}_0 = \Pi_0 = \Phi(0)$ and $\hat{\Pi}_{n+m} = \Pi_n = \Phi(1)$. Consider the polynomial function of degree $n+m$, $\Gamma \colon I \to \mathbb{R}^{n+m}$, defined by:

$$\Gamma(x) := (\Gamma_1(x), ..., \Gamma_{n+m}(x))^T := (\Phi(x), x^{n-1}, ..., x^{n+m-2})^T. \tag{4.6}$$

It is nondegenerate, in the sense that $(\mathbf{1}, \Gamma_1, ..., \Gamma_{n+m})$ is a basis of $\mathscr{P}_{n+m}$, or, equivalently, that its image spans $\mathbb{R}^{n+m}$. Therefore, blossoms in the space $\mathscr{P}_{n+m}$ can be obtained geometrically from Γ as follows. Any polynomial function F of degree less than or equal to $n+m$ can be uniquely written as the image $F = h \circ \Gamma$ of Γ under an affine map. The blossom f of F then satisfies $f = h \circ \gamma$, where γ is the ordinary blossom of the polynomial function Γ. As for γ, it satisfies

$$\{\gamma(x_1, ..., x_{n+m})\} = \bigcap_{i=1}^{r} \operatorname{Osc}_{n+m-\mu_i} \Gamma(\tau_i), \tag{4.7}$$

for all $(n+m)$-tuples $(x_1, ..., x_{n+m})$ which are equal to $(\tau_1{}^{\mu_1}, ..., \tau_r{}^{\mu_r})$ up to a permutation. In particular, if $\vartheta: \mathbb{R}^{n+m} \to \mathbb{R}^n$ denotes the projection onto the first n components, we have:

$$\Phi = \vartheta \circ \Gamma, \qquad \hat{\varphi} = \vartheta \circ \gamma. \tag{4.8}$$

Let us introduce the polynomial function $G: I \to \mathbb{R}^{n+m}$ of degree less than or equal to $n+m-2$ obtained by fixing two variables in the blossom γ of Γ as follows:

$$G(x) := \gamma(0, 1, x^{n+m-2}), \qquad x \in I. \tag{4.9}$$

The ordinary blossom g is given by:

$$g(x_1, ..., x_{n+m-2}) = \gamma(0, 1, x_1, ..., x_{n+m-2}). \tag{4.10}$$

By its very definition, we have

$$\{G(x)\} = \mathrm{Osc}_{n+m-1}\, \Gamma(0) \cap \mathrm{Osc}_{n+m-1}\, \Gamma(1) \cap \mathrm{Osc}_2\, \Gamma(x) \quad \text{for } x \neq 0, 1, \tag{4.11}$$

while $G(0) = \gamma(0^{n+m-1}, 1)$, $G(1) = \gamma(0, 1^{n+m-1})$, i.e.,

$$\begin{aligned} \{G(0)\} &= \mathrm{Osc}_1\, \Gamma(0) \cap \mathrm{Osc}_{n+m-1}\, \Gamma(1), \\ \{G(1)\} &= \mathrm{Osc}_{n+m-1}\, \Gamma(0) \cap \mathrm{Osc}_1\, \Gamma(1). \end{aligned} \tag{4.12}$$

We can thus write

$$G(x) = \Gamma(x) + \lambda(x)\, \Gamma'(x) + \mu(x)\, \Gamma''(x), \qquad x \in I, \tag{4.13}$$

where, due to (4.12) and to the linear independence of the derivatives of Γ, $\mu(0) = \mu(1) = 0$. Furthermore, for $x \in \,]0, 1[$, the two quantities $\lambda(x)$ and $\mu(x)$ are calculated so as to ensure that $G(x) \in \mathrm{Osc}_{n-1}\, \Gamma(0) \cap \mathrm{Osc}_{n-1}\, \Gamma(1)$. Now, one can easily check that $X_{n-1} = 0$ (resp. $X_n = 0$) is a necessary and sufficient condition for a point $X := (X_1, ..., X_{n+m}) \in \mathbb{R}^{n+m}$ to belong to the osculating hyperplane $\mathrm{Osc}_{n+m-1}\, \Gamma(0)$ (resp. $\mathrm{Osc}_{n+m-1}\, \Gamma(1)$). Therefore, the $(n-1)$-th and n-th components of $G(x)$ are equal to zero. Now, let us consider the polynomial function $F: I \to \mathbb{R}^n$, of degree less than or equal to $n+m-2$, defined by

$$F(x) := \vartheta \circ G(x). \tag{4.14}$$

Since $\Phi = \vartheta \circ \Gamma$, it follows from (4.13) and (4.14) that:

$$F(x) = \Phi(x) + \lambda(x)\, \Phi'(x) + \mu(x)\, \Phi''(x). \tag{4.15}$$

Again, one can check that $X_{n-1}=0$ (resp. $X_n=0$) is a necessary and sufficient condition for a point $X:=(X_1, ..., X_n) \in \mathbb{R}^n$ to belong to the osculating hyperplane $\mathrm{Osc}_{n-1}\,\Phi(0)$ (resp. $\mathrm{Osc}_{n-1}\,\Phi(1)$). Now, the $(n-1)$-th and n-th components of $G(x)$ being equal to zero, it follows from (4.14) that the $(n-1)$-th and n-th components of $F(x)$ are also equal to zero. Consequently,

$$F(x) \in \mathrm{Osc}_{n-1}\,\Phi(0) \cap \mathrm{Osc}_{n-1}\,\Phi(1) \cap \mathrm{Osc}_2\,\Phi(x) \quad \text{for } x \neq 0, 1. \tag{4.16}$$

Since the right hand side of (4.16) is reduced to the single point $\varphi(0\,,1\,,x^{n-2})$, F eventually proves to be the quasi-Chebyshev function of order $n-2$ obtained by fixing two variables in the blossom φ of Φ as follows:

$$F(x) = \varphi(0\,,1\,,x^{n-2}). \tag{4.17}$$

According to the subblossoming principle, as a quasi-Chebyshev function of order $n-2$ its blossom f is given by

$$f(x_1, ..., x_{n-2}) = \varphi(0, 1, x_1, ..., x_{n-2}), \tag{4.18}$$

and its Chebyshev-Bézier points w.r. to $(0, 1)$ are therefore the points

$$f(0^{n-2-i}, 1^i) = \varphi(0^{n-1-i}, 1^{i+1}) = \Pi_{i+1}, \qquad i=0, ..., n-2. \tag{4.19}$$

The last two components of $F(x)$ being equal to 0, function F is actually a polynomial function of degree less than or equal to $n-2$ with values in $\mathbb{R}^{n-2}$. Its blossom f is also the ordinary blossom for such functions, and its Chebyshev-Bézier points are its Bézier points. On the other hand, on account of (4.14), (4.9), and (4.8), function F can be written as:

$$F(x) = \hat{\varphi}(0\,,1\,,x^{n+m-2}). \tag{4.20}$$

Through this latter equality, F is now being viewed as a polynomial function of degree less than or equal to $n+m-2$. The corresponding blossom $\hat{f}$ of F is given by $\hat{f}(x_1, ..., x_{n+m-2}) := \hat{\varphi}(0, 1, x_1, ..., x_{n+m-2})$, and its Bézier points (w.r. to $(0, 1)$) are:

$$\hat{f}(0^{n+m-2-i}, 1^i) = \hat{\varphi}(0^{n+m-1}, 1^{i+1}) = \hat{\Pi}_{i+1}, \qquad i=0, ..., n+m-2. \tag{4.21}$$

This implies that the Bézier points $\hat{\Pi}_1, ..., \hat{\Pi}_{n+m-1}$ of F considered as an element of $\mathscr{P}^{n-2}_{n+m-2}$ are obtained by applying a degree elevation process from degree $n-2$ to degree $n+m-2$ starting from the Bézier points $\Pi_1, ..., \Pi_{n-1}$ of $F \in \mathscr{P}^{n-2}_{n-2}$. ∎

From Proposition 4.1 we can derive the expression of the Chebyshev-Bernstein basis, that is to say, of the optimal normalized totally positive basis of the space $\mathscr{E}(\Phi)$.

COROLLARY 4.2. *Let us denote by* $(B_0^{n+m}, ..., B_{n+m}^{n+m})$ *the Bernstein basis of degree* $n+m$. *Then, the Chebyshev-Bernstein basis in the space* $\mathscr{E}(\Phi)$ *associated with the quasi-Chebyshev function* Φ *defined in* (4.3) *is given by*:

$$\mathscr{B}_0(x)=B_0^{n+m}(x), \qquad \mathscr{B}_n(x)=B_{n+m}^{n+m}(x),$$
$$\mathscr{B}_i(x)=\sum_{\ell=i}^{i+m} \frac{\binom{\ell-1}{i-1}\binom{n+m-\ell-1}{n-i-1}}{\binom{n+m-2}{m}} B_\ell^{n+m}(x), \qquad i=1, ..., n-1,\ x\in[0,1]. \tag{4.22}$$

Proof. Let $P_0, ..., P_n$ be the Bézier points of a polynomial function $F\in\mathscr{P}_n^d$. Then, the Bézier points of F considered as an element of $\mathscr{P}_{n+1}^d$ are given by the well-known degree elevation formulae:

$$P_0^{n+1}=P_0^n,$$
$$P_i^{n+1}=\frac{i}{n+1}P_{i-1}^n+\Big(1-\frac{i}{n+1}\Big)P_i^n \quad i=1, ..., n, \qquad P_{n+1}^{n+1}=P_n^n. \tag{4.23}$$

By iteration of (4.23), it is easy to check that the Bézier points of the same function F viewed as an element of $\mathscr{P}_{n+p}^d$ are, with the convention that $\binom{q}{r}=0$ whenever $q<r$:

$$P_i^{n+p}=\sum_{k=0}^{n}\frac{\binom{i}{k}\binom{n+p-i}{n-k}}{\binom{n+p}{p}}P_k^n, \qquad i=0, ..., n+p. \tag{4.24}$$

According to Proposition 4.1, the Bézier points $\hat{\Pi}_1, ..., \hat{\Pi}_{n+m-1}$ therefore satisfy:

$$\hat{\Pi}_i=\sum_{k=1}^{n-1}\frac{\binom{i-1}{k-1}\binom{n+m-i-1}{n-k-1}}{\binom{n+m-2}{m}}\Pi_k, \qquad i=1, ..., n+m-1. \tag{4.25}$$

Now, we know that

$$\Phi(x)=\sum_{i=0}^{n}\mathscr{B}_i(x)\,\Pi_i=\sum_{\ell=0}^{n+m}B_\ell^{n+m}(x)\,\hat{\Pi}_\ell, \qquad x\in[0,1]. \tag{4.26}$$

Let us replace $\hat{\Pi}_0$ by Π_0, $\hat{\Pi}_{n+m}$ by Π_n, and the other $\hat{\Pi}_i$'s by (4.25) in the right hand side of (4.26). Due to the linear independence of the points $\Pi_0, ..., \Pi_n$, we obtain (4.22) by identification. ∎

For a fixed integer $n \geqslant 2$, let us set $\Phi^m(x) := (x, ..., x^{n-2}, x^{n+m}, (1-x)^{n+m})^T$, and consider $m \in \mathbb{N}$ as a parameter. Given $n+1$ fixed points $P_0, ..., P_n \in \mathbb{R}^d$, let F^m be the $\mathscr{E}(\Phi^m)$-function with Chebyshev-Bézier points are $P_0, ..., P_n \in \mathbb{R}^d$. Denoting by $\hat{P}_1^m, ..., \hat{P}_{n+m-1}^m$ the points obtained by m consecutive degree elevation processes from $P_1, ..., P_{n-1}$, this function F^m is also the polynomial function of degree less than or equal to $n+m$ with Bézier points $\hat{P}_0^m, ..., \hat{P}_{n+m}^m$, where $\hat{P}_0^m := P_0$, $\hat{P}_{n+m}^m := P_n$. Therefore, the curve defined by F^m can be drawn by applying the classical de Casteljau or subdivision algorithm starting from the points $\hat{P}_i^m$.

The parameter m acts as a shape parameter. For $m=0$, we obtain the polynomial curve of degree n with $P_0 ... P_n$ as its control polygon. When m goes to ∞, the polygon $\hat{P}_1^m \cdots \hat{P}_{n+m-1}^m$ converges towards the polynomial curve $\mathscr{C}$ of degree $n-2$ with $P_1 \cdots P_{n-1}$ as its control polygon. Hence the limit position of the curve defined by F^m will be composed of $\mathscr{C}$ and the two segments $P_0 P_1$ and $P_{n-1} P_n$. This is illustrated in Figs. 1 and 2.

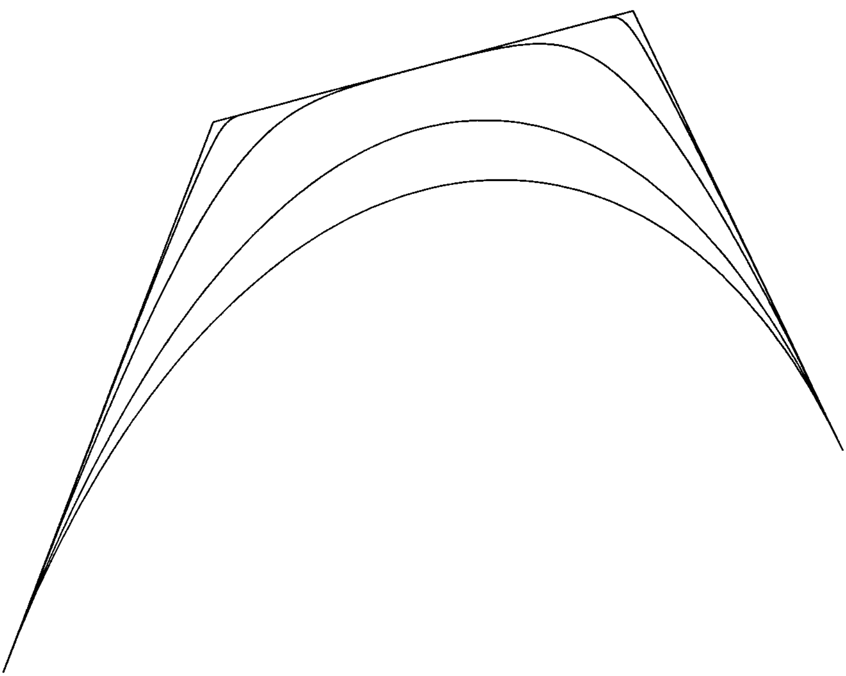

FIG. 1. Curves in the space spanned by $1, x, x^{3+m}, (1-x)^{3+m}$ for $m = 0; 1; 10; 100$.

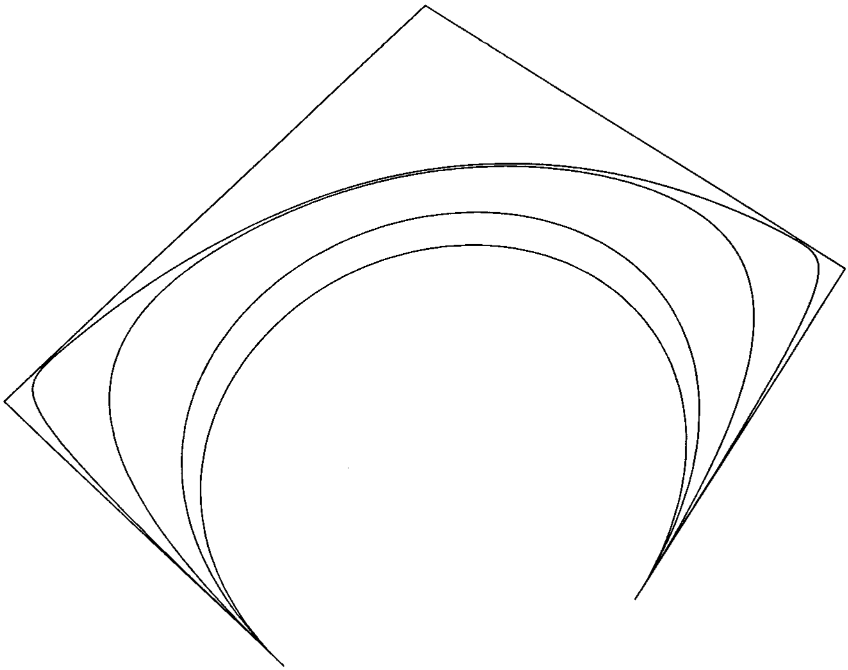

FIG. 2. Curves in the space spanned by $1, x, x^2, x^{4+m}, (1-x)^{4+m}$ for $m = 0; 1; 10; 100$.

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